3E4: Modelling Choice

Lecture 6

- *Goal Programming*
- *Multiple Objective Optimisation*
- *Portfolio Optimisation*

Announcements

Supervision 2

- To be held by the end of <u>next</u> week
- Present your solutions to all Lecture 4, 5 and 6 homeworks

3E4 : Lecture Outline

Lecture 1. Management Science & Optimisation Modelling: Linear Programming

Lecture 2. LP: Spreadsheets and the Simplex Method

Lecture 3. LP: Sensitivity & shadow prices
Reduced cost & shadow price formulae

Lecture 4. Integer LP: branch & bound

Lecture 5. Network flows problems

Lecture 6. Multiobjective LP

Lecture 7 – 8. Introduction to nonlinear programming

Hard *versus* ***soft*** *constraints*

- **Hard constraints** are constraints that cannot be violated
- In some cases, hard constraints are too restrictive...
  - You have a maximum price <u>in mind</u> when buying a car
- An alternative is given by **soft constraints**
- They represent **goals** or **targets** that we'd like to achieve
- **Goal Programming** gives a way to handle (several) soft constraints

A Goal Programming Example: Hotel Expansion

- A hotel director wants to expand the convention centre at his hotel
- The types of conference rooms being considered are:

	Size (sq ft)	Unit Cost
Small	400	£18,000
Medium	750	£33,000
Large	1,050	£45,150

- He would like to add 5 small, 10 medium and 15 large conference rooms.
- He would also like the total expansion to be 25,000 square feet and to limit the cost to £1,000,000

Decision Variables

X_1 = number of small rooms to add

X_2 = number of medium rooms to add

X_3 = number of large rooms to add

The Goals

- Goal 1: The expansion should include *not many fewer than* 5 small conference rooms.
- Goal 2: The expansion should include *not many fewer than* 10 medium conference rooms.
- Goal 3: The expansion should include *not many fewer than* 15 large conference rooms.
- Goal 4: The expansion should consist of *approximately* 25,000 square feet.
- Goal 5: The expansion should *not* cost *much more than* £1,000,000.

Defining the Goal Constraints

- Small rooms: $X_1 \geq 5 - u_1$
- Medium rooms: $X_2 \geq 10 - u_2$
- Large rooms: $X_3 \geq 15 - u_3$
- Expansion:

$$400\ X_1 + 750\ X_2 + 1050\ X_3 = 25000 + o_4 - u_4$$

- Total Cost:

$$18\ X_1 + 33\ X_2 + 45.15\ X_3 \leq 1000 + o_5$$

- Deviational variables: Model
  - "not much more" using **undershoot** variable $u_i \geq 0$
  - "not much less" using **overshoot** variable $o_i \geq 0$

GP Objective Functions

- There are several objective functions we could formulate for a GP problem.
- Minimize the sum of the deviations

$$\text{MIN } u_1+u_2+u_3+(o_4+u_4)+ o_5$$

- Difficulty: The deviations measure different things, so what does this objective represent?

Percentage Deviations

- Minimize the sum of **percentage deviations**

$$\text{MIN } u_1/t_1+u_2/t_2+u_3/t_3+u_4/t_4+o_4/t_4+o_5/t_5$$

where t_i represents a given **target** value of goal i

- Difficulty arises in comparing the following deviations from targets:
  - Suppose we underachieve goal 1 by 1/5=20%
  - Suppose we exceed goal 5 by 200,000/1,000,000= 20%
  - Both deviations are 20%. But is being £200,000 over budget just as undesirable as having one too few small rooms?
  - Only the **<u>decision maker</u>** can say for sure.

Weighting the goals

- **Weights** can be used in the previous objectives to allow the decision maker to indicate the relative importance of various goals

$$\text{MIN } \mathbf{w_1}u_1/t_1+\mathbf{w_2}u_2/t_2+\mathbf{w_3}u_3/t_3+\mathbf{w_4}u_4/t_4+\mathbf{v_4}o_4/t_4+\mathbf{v_5}o_5/t_5$$

- Initially, we will assume all weights v_i,w_i equal 1
- The decision maker can change the weights interactively with the model

Comments About GP

- GP involves making trade-offs among the goals until the most satisfying solution is found
- GP objective function values should not be compared because the weights are changed in each iteration. Compare the solutions!
- A very large weight will effectively change a soft constraint to a hard constraint.
- A weight of zero will effectively delete a constraint
- Hard constraints can be placed on deviational variables.

The MiniMax Objective

- The minimax objective

  $$\min \max_i \{v_i o_i/t_i\ ,\ w_i u_i/t_i\}$$

  is used to minimize the biggest percentage deviation from any goal.
- Minimize weighted maximum percentage deviation

$$\text{Min: } \alpha$$

subject to: $v_i o_i/t_i \le \alpha$

$w_i u_i/t_i \le \alpha$

plus other constraints

Exercise: *Hotel Expansion*

- What is the goal program using minimax percentage deviations?
- Min α subject to:
  - Small rooms: $X_1 \ge 5 - u_1$
  - Medium rooms: $X_2 \ge 10 - u_2$
  - Large rooms: $X_3 \ge 15 - u_3$
  - Area: $400\,X_1 + 750\,X_2 + 1050\,X_3 = 25000 + o_4 - u_4$
  - Cost: $18\,X_1 + 33\,X_2 + 45.15\,X_3 \le 1000 + o_5$
  - all deviational vars. nonnegative ($u_1, \ldots, u_4, o_4, o_5 \ge 0$)

    $\alpha \ge$ all weighted percentage deviations, i.e.

    $\alpha \ge w_1u_1/t_1, w_2u_2/t_2\ , w_3u_3/t_3\ , w_4u_4/t_4\ , v_4o_4/t_4\ , v_5o_5/t_5$

    where $w_1, \ldots, w_4u, v_4, v_5$ are positive parameters

Summary of Goal Programming

1. Formulate decision variables and hard constraints in the usual way.
2. State the goals of the problem along with their target values.
3. Create additional constraints that would achieve the goals exactly.
4. Transform the latter constraints into goal constraints by including deviational variables for undesirable deviations
5. Formulate an objective that penalizes the undesirable deviations.
6. Identify appropriate weights for the objective.
7. Solve the problem.
8. Inspect the solution to the problem. If the solution is unacceptable, return to step 6 and revise the weights as needed.

Multiple Objectives

- **Multiple objectives** are common
  - Maximal Return and Minimal Risk
  - Maximal Profit and Minimal Pollution
- In GP we have an objective for each goal:
  - Minimize the undesirable deviation from the target value
- Objectives often **conflict** with one another

Multiple Objective Linear Programming (MOLP)

- An MOLP problem is an LP problem with more than one objective function.
- MOLP problems can be viewed as special types of GP problems where we must also determine target values for each goal or objective.

An MOLP Example: A Mining Company

- A mining company operates two coal mines
- Monthly production by a shift of workers at each mine is summarized as follows:

Type of Coal	Mine 1	Mine 2
High-grade	12 tons	4 tons
Medium-grade	4 tons	4 tons
Low-grade	10 tons	20 tons
Cost per month	£40,000	£20,000
Litres of toxic water produced	600	1800

- Additional total demand over the next year of 48 tons for high-grade, 28 tons for medium-grade, and 100 tons for low-grade coal needs to be satisfied by scheduling an extra shift at one or both mines

Variables and Objectives

- Variables: X_i = number of months to schedule an extra shift at mine *i*

- There are two objectives:
  - Min: £40 X_1 + £20 X_2 } Production costs
  - Min: 600 X_1 + 1800 X_2 } Toxic water

Defining the Constraints

- High-grade coal required
  $12 X_1 + 4 X_2 \geq 48$
- Medium-grade coal required
  $4 X_1 + 4 X_2 \geq 28$
- Low-grade coal required
  $10 X_1 + 20 X_2 \geq 100$
- Nonnegativity conditions
  $X_1, X_2 \geq 0$
- Production takes place in the next year
  $X_1, X_2 \leq 12$

Handling Multiple Objectives

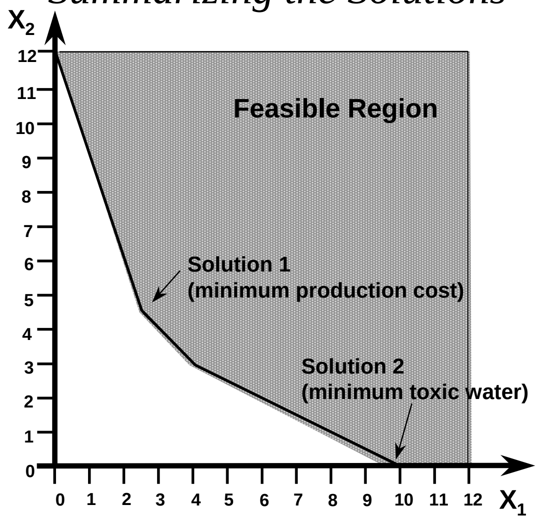
- If management gives us target values for the objectives then we can treat them like goals:

Goal 1: The total cost of productions should not exceed t_1 by much.

Goal 2: The amount of toxic water produce should not exceed t_2 by much

- Alternatively, we can solve two separate LP problems, independently optimising each objective, to find values for t_1 and t_2.

Summarizing the Solutions



Solution	X_1	X_2	Cost	Toxic Water
1	2.5	4.5	£190,000	9,600
2	10	0	£400,000	6,000

Defining The Goals

- **Goal 1**: The total cost of productions should not be much more than £190,000
- **Goal 2**: We should not produce much more than 6,000 litres of toxic water

Defining an Objective

- We can minimize the sum of % deviations as follows:

$$\text{MIN: } w_1\left(\frac{(40X_1 + 20X_2) - 190}{190}\right) + w_2\left(\frac{(600X_1 + 1800X_2) - 6000}{6000}\right)$$

- (Where are the overshoots and undershoots??)
- This is just a linear combination of the decision variables X_1 and X_2
- The resulting problem for given weights is an LP
- Solver produces optimal solutions at corner points of the feasible region, no matter what weights are used.

Visualising the objectives

In this particular example

- we have two objectives (cost, water) which are determined by our two variables (x_1,x_2)
- we can transform our variables into functions of our objectives
- we can therefore view all objective pairs (cost, water) that it is possible to achieve by feasible decision pairs (x_1,x_2)

This will be useful in explaining what MOLP and GP are actually doing.

Changing Coordinates

- We have two objectives

  $u = 4x_1 + 2x_2$ (cost in £10,000)

  $v = 6x_1 + 18x_2$ (toxic water in 100 ltr)

- If we solve the equations we obtain

  $x_1 = 3u/10 - v/30$

  $x_2 = -u/10 + v/15$

- We can use this to express the constraints in (u,v) variables

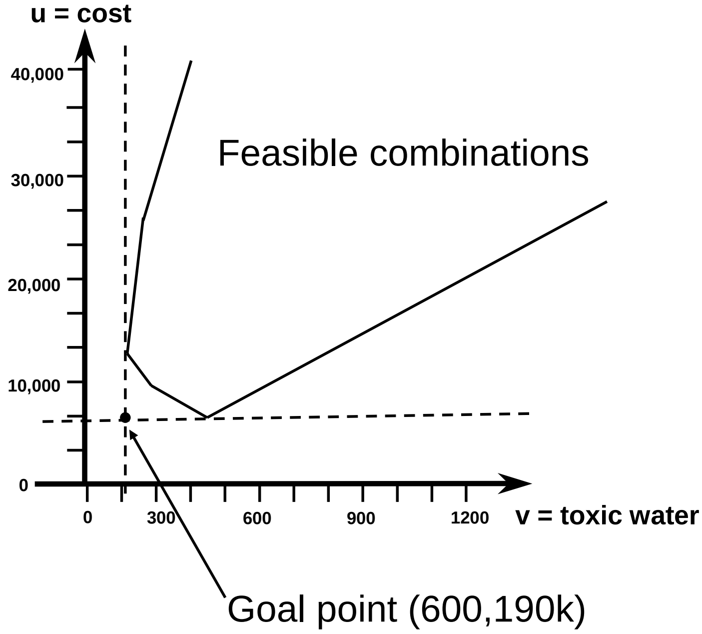
Constraints in objective variables

$$\begin{aligned}
16u/5 - 2v/15 &\geq 48 \quad \text{(HG coal)}\\
4u/5 + 2v/15 &\geq 28 \quad \text{(MG coal)}\\
u + v &\geq 100 \quad \text{(LG coal)}\\
0 \leq 3u/10 - v/30 &\leq 12 \quad (0 \leq x_1 \leq 12)\\
0 \leq -u/10 + v/15 &\leq 12 \quad (0 \leq x_2 \leq 12)
\end{aligned}$$

Simplified

$$\begin{aligned}
24u - v &\geq 360\\
6u + v &\geq 210\\
u + v &\geq 100\\
0 \leq 9u - v &\leq 12\\
0 \leq -3u + 2v &\leq 12
\end{aligned}$$

Let’s draw a picture of the feasible set in (u,v) space



Visualizing the objectives: an easier graphical approach

- Given the MOLP

  Max $z_1=c_{11}x_1+ \ldots + c_{1n}x_n$

  …

  Max $z_m=c_{m1}x_1+ \ldots + c_{mn}x_n$

  s.t. $Ax=b$

  $x \geq 0$
- Draw the feasible region.
- Evaluate all the corner points of the feasible region, say $x^{(1)},\ldots,x^{(K)}$

Visualizing the objectives: an easier graphical approach

- Then calculate the corresponding transformed points

$$z^{(1)}=Cx^{(1)},\ldots,\ z^{(K)}=Cx^{(K)}$$

  where C is the matrix $[c_{ij}]$
- Draw the objective space as the polyhedron having $z^{(1)},\ldots,\ z^{(K)}$ as corner points.

Example

$$\begin{aligned}
\max\ z_1 &= 3x_1 - x_2 \\
\max\ z_2 &= -2x_1 + 4x_2 \\
\text{s.t.}\quad & -x_1 + x_2 \leq 5 \\
& x_1 + 2x_2 \leq 12 \\
& x_1 \leq 8 \\
& x_1,\ x_2 \geq 0
\end{aligned}$$

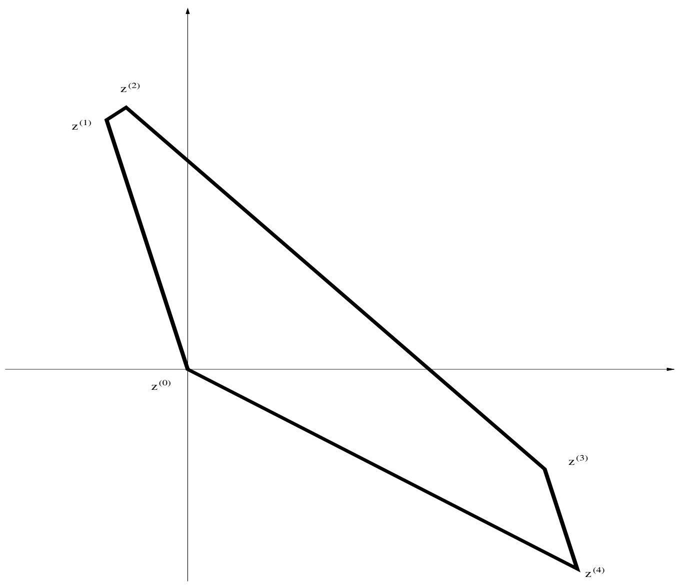
Example

max $z_1 = 3x_1 - x_2$
max $z_2 = -2x_1 + 4x_2$
s.t. $-x_1 + x_2 \leq 5$
$x_1 + 2x_2 \leq 12$
$x_1 \leq 8$
$x_1, x_2 \geq 0$

Feasible region corner points
$x^{(0)}=(0,0)$; $x^{(1)}=(0,5)$;
$x^{(2)}=(0.66,5.66)$;
$x^{(3)}=(8,2)$; $x^{(4)}=(8,0)$.

Objective space corner points
$z^{(0)}=(0,0)$; $z^{(1)}=(-5,20)$; $z^{(2)}=(-3.66,21.33)$;
$z^{(3)}=(22,-8)$; $z^{(4)}=(24,-16)$.

Objective space



A new concept

Pareto optimality

- Since we are maximizing, we (strictly) prefer $(z_1^{\#}, z_2^{\#})$ to (z_1, z_2) if

$$z_1^{\#} \geq z_1 \text{ and } z_2^{\#} \geq z_2$$

and

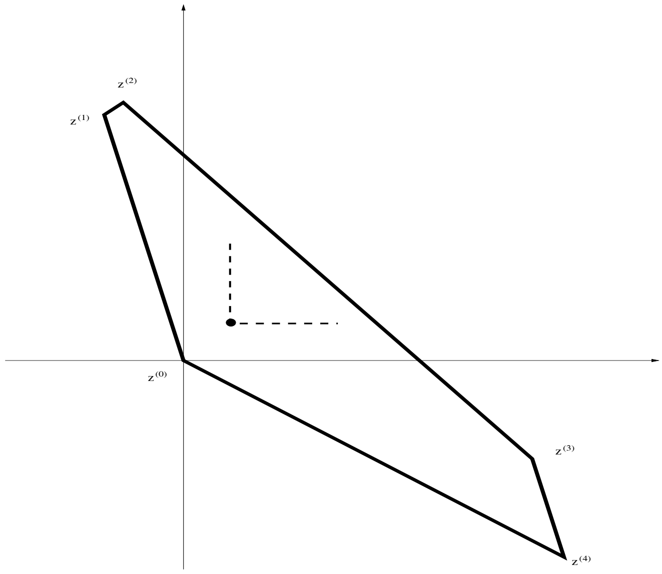
at least one inequality holds strictly

- A solution to a multi-objective optimisation problem is called **Pareto optimal** (or **efficient**) if there is no other feasible solution which is preferred.

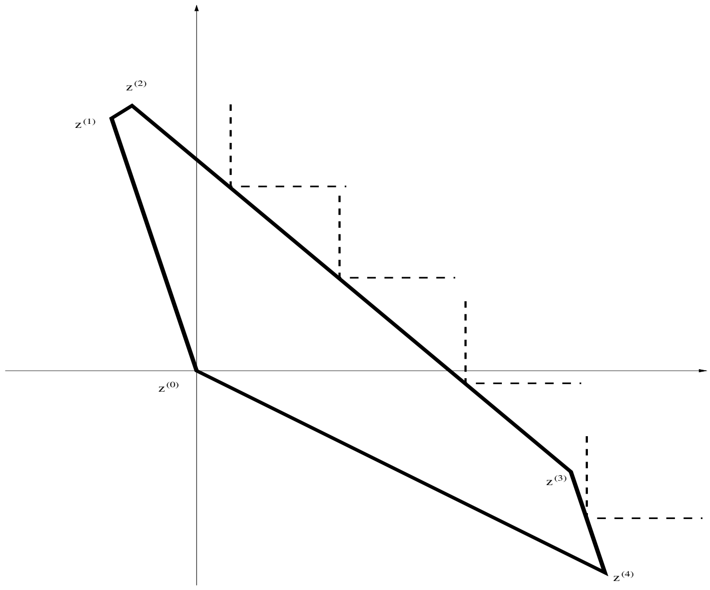
The efficient frontier

- The set of all Pareto optimal strategies in the objective space is called the efficient frontier
- Goal programming amounts to finding a point on the efficient frontier that is as close as possible to the "goal point" = vector of targets
- The "distance" function is user defined by choice of weights
- In principle, changing weights in the objective allows us to "trace" the efficient frontier. In practise, **`Solver`** will only give us corner points because it uses the Simplex Method.

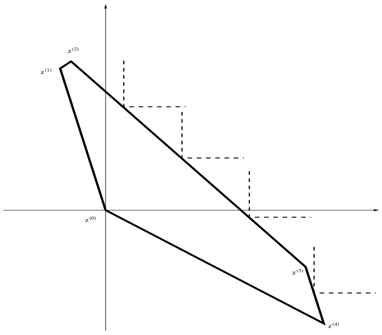
Finding efficient points



Finding efficient points

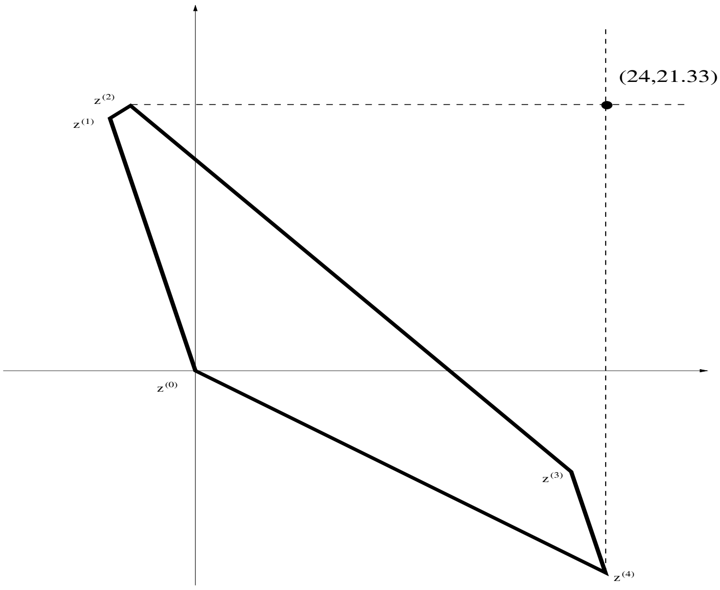


Finding efficient points



- The efficient frontier is made up of all points lying along the segments joining points $z^{(2)}$ -$z^{(3)}$ and $z^{(3)}$ - $z^{(4)}$
- The efficients points are therefore all points lying along the segments joining points $x^{(2)}$ -$x^{(3)}$ and $x^{(3)}$ - $x^{(4)}$

Goal point



The MiniMax Objective again

$$\min \ \alpha$$

$$\text{subject to } \ w_1\left(\frac{(40X_1 + 20X_2) - 190}{190}\right) \le \alpha$$

$$w_2\left(\frac{(600X_1 + 800X_2) - 6000}{6000}\right) \le \alpha$$

plus other constraints

By changing the weights we can trace the complete efficient frontier

A formal statement on

Positive weights and Pareto optima

If

- we are maximizing K objectives, $v_1, \dots, v_K$
- we have K positive weights $w_1, \dots, w_K$
- $(v_1^\#, \dots, v_K^\#)$ is an optimal solution of Min $w_1v_1 + \dots + w_kv_K$ over all feasible combinations $(v_1, \dots, v_K)$.

Then $(v_1^\#, \dots, v_K^\#)$ is Pareto optimal.

Proof Let $(v_1, \dots, v_K)$ represent the payoffs of some feasible strategy. Assume for a contradiction that $v_k \geq v_k^\#$ for $k=1, \dots, K$, with strict inequality for some index $k=i$. Then

$$w_kv_k \geq w_kv_k^\# \text{ for all } k \text{ (since } w_k \text{ nonnegative)}$$

with strict inequality for $k = i$ (since w_k positive), hence

$$\Sigma\, w_kv_k > \Sigma\, w_kv_k^\#.$$

The last inequality contradicts global optimality of $(v_1^\#, \dots, v_K^\#)$ for the weighted sum problem. **QED**

Comments About MOLP

- Goals for a Multi-Objective problem can be established by solving the problem once for each objective in turn.
- Solutions obtained using the MiniMax or weighted sum objective with positive weights are Pareto Optimal.
- The percentage deviation is
  - (actual – target)/target for minimization objectives
  - (target – actual)/target for maximization objectives
- If a target value is zero, use the weighted deviations rather than weighted % deviations.

An application of practical importance

Financial Portfolio Optimisation

- Investment instrument: single stock or other investment with known “statistical behaviour” (e.g. average return and variance in last 3 years)
- Investment portfolio: mix of investment instruments
- There is a trade-off between expected return and risk of an investment portfolio

Portfolio of three instruments

- Expected returns r_1, r_2 and r_3 (in %)
- x_i: % of the portfolio in instrument i = 1,2,3 ($x_1 + x_2 + x_3 = 100\%$)
- Expected return of portfolio

$$E = x_1 r_1 + x_2 r_2 + x_3 r_3$$

- Expected return if you invest £ M is ExM

Risk and Variance

- Variances of the instruments are σ_1^2, σ_2^2 , σ_3^2
- Variance is often used as a **measure of risk**
- Covariance σ_{12} (= σ_{21}) between price of stock 1 and stock 2, for example, is a measure of the degree to which one return goes up or down when the other goes up or down
- Covariances σ_{ik} are stored in the 3x3 covariance matrix Q ($\sigma_{ii}=\sigma_i^2$)
- **COVAR** function in Excel allows you to estimate covariances using historical data

Risk of a portfolio

- Formula for the variance of the portfolio is

$$\sigma_P^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + x_3^2\sigma_3^2 + 2x_1x_2\sigma_{12} + 2x_1x_3\sigma_{13} + 2x_2x_3\sigma_{23}$$

- For any number of investment instruments, the formula is easily remembered as

$$\sigma_P^2 = x^T Q x$$

where x is the (column) vector of percentages of the portfolio in the various instruments

- Dimension of x = number of instruments in the portfolio
- Variance formula is **nonlinear** in x, in fact quadratic

Dealing with the trade-off

- We wish to determine the split x of instruments in the portfolio
- We want to
  - maximize expected return
  - minimize risk

3 Stock Investment Example

- 3 stocks to consider

Investment	Expected Return	Covariance Matrix Q		
Stock A	0.076417	0.00258	-0.00025	0.00440
Stock B	0.134333	-0.00025	0.00276	-0.00542
Stock C	0.149333	0.00440	-0.00542	0.03677

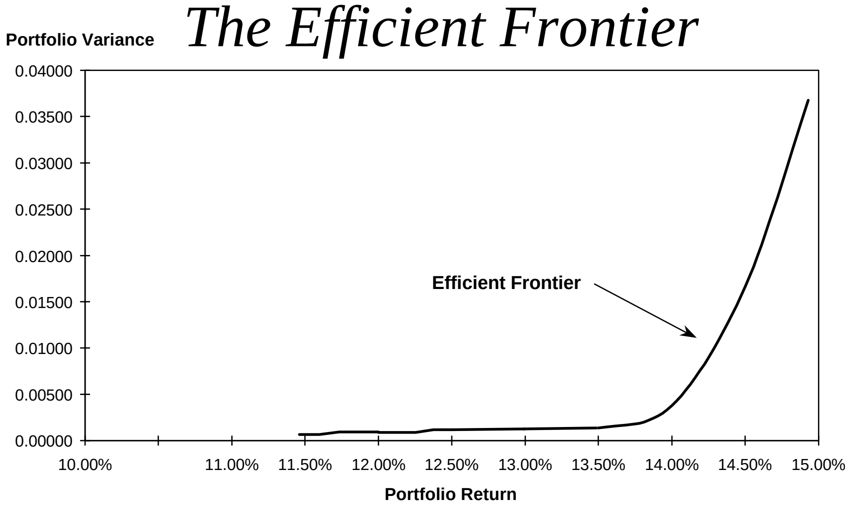
- For a given split of capital,
  e.g. $x_1 = 0.3$, $x_2 = 0.4$, $x_3 = 0.3$,
  - expected return
    $= 0.076417\ x_1 + 0.134333\ x_2 + 0.149333\ x_3$
    $= 12.15\%$
  - variance is $x^TQx = 0.003413$,
    where $x = (x_1, x_2, x_3)$, Q = covariance matrix

What risk is acceptable?

- Constraint approach:
  - set the level of expected return and then minimize the risk
  - alternatively: set level of risk and maximize expected return
- Depicting the efficient frontier: Plot the minimal portfolio risk associated with each possible level of return

3 Stock Investment Example

The Efficient Frontier



Multiple Objective Approach

- Choose weight w between 0 and 1 and maximize

  (1-w)*Expected return - w*Variance

- As you vary w between 0 and 1 you trace the efficient frontier
- Risk averse investor prefers large w (near 1)

Today's key points

- MOP deals with multiple objectives
- GP is one way of solving MOP
- Aim in MOP is to help decision maker to find a suitable point on the efficient frontier
- One possibility is to optimise a weighted sum of all objectives
- Weighted MiniMax is an alternative approach
- Different weights lead to different points on the efficient frontier
- Suitable weights need to be determined by the decision maker in interaction with the model

Lecture 6 *3E4 Homework*

1. (From 2001 exam) A tract of land of 1000 acres, owned by a local government council, is used as a bird sanctuary, for sheep grazing, and for recreation (walking). The council is reviewing the uses of this land and has designed indices to show the benefits of its use. All 1000 acres contribute to bird life. The bird index is 2 for each acre not used for grazing or recreation, 1 for each acre used for either grazing or recreation but not both, and 0 for each acre jointly used for grazing and recreation. The grazing index is 2 for each acre used for grazing but not recreation, 1 for each acre used for both grazing and recreation, and 0 otherwise. The recreation index is 2 for each acre used for recreation, 0 otherwise. After consulting community groups and other stakeholders, the council sets the following goals: the bird, grazing and recreation indices should exceed 3000, 1000 and 1000 respectively.

Formulate (mathematically) a goal program for this problem using percentage deviations.

Lecture 6 *3E4 Homework*

2. Formulate and solve, in Excel, the minimax goal program associated with the bi-objective 3 Stock Investment Example.

- use one of the spreadsheets on website as a starting point
  - you will first have to define the targets of your goals with Excel.
- make the weights data cells in your spreadsheet
- Answer
  - Target for expected return = 14.93%
  - Target for variance = 0.00110
  - Optimal split for equal weights is x = (0.121,0.744, 0.134)

Lecture 6 *3E4 Homework*

3. Consider a bi-objective LP, with goals Max u and Max v which are characterised by the feasible set

$$0 \le u \le 3,\ 0 \le v \le 3,\ u+v \le 4.$$

3.a. Sketch the feasible region and show the Pareto optimal set (efficient frontier)

3.b. Give nonnegative weights w_1, w_2 such that the problem

$$\text{Max } w_1 u + w_2 v \text{ subject to above constraints}$$

has a non-efficient solution. What solution is this?