3E4 Supervision 2

• should be held between March 2 and March 12

• students will present solutions to all homework questions from Lectures 4, 5 and 6

Lecture 4 Homework & SOLUTIONS

1.a. Use binary variables to model the capital budgeting problem on the next slide.

1.b. Use the binary variables defined in 1.a. to model each of the additional logical conditions (which need not be used simultaneously):
   – Of projects 1, 3 & 6, no more than one may be selected
   – Of projects 1, 3 & 6, exactly one must be selected
   – Project 4 cannot be selected unless project 5 is also selected
Lecture 4 Q1: Capital Budgeting Problem

<table>
<thead>
<tr>
<th>Project</th>
<th>Expected NPV (in £000s)</th>
<th>Capital (in £000s) Required in Year</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>£141</td>
<td>£75</td>
<td>£25</td>
<td>£20</td>
<td>£15</td>
<td>£10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>£187</td>
<td>£90</td>
<td>£35</td>
<td>£0</td>
<td>£0</td>
<td>£30</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>£121</td>
<td>£60</td>
<td>£15</td>
<td>£15</td>
<td>£15</td>
<td>£15</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>£83</td>
<td>£30</td>
<td>£20</td>
<td>£10</td>
<td>£5</td>
<td>£5</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>£265</td>
<td>£100</td>
<td>£25</td>
<td>£20</td>
<td>£20</td>
<td>£20</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>£127</td>
<td>£50</td>
<td>£20</td>
<td>£10</td>
<td>£30</td>
<td>£40</td>
<td></td>
</tr>
</tbody>
</table>

The company currently has £250,000 available to invest in new projects. It has budgeted £75,000 for continued support for these projects in year 2 and £50,000 per year for years 3, 4, and 5.

Solution to Lecture 4 Homework, Q.1.a.

Decision Variables and Objective

- Decision Variables:
  \( X_i = 1 \) if project is selected
  \( X_i = 0 \) otherwise, \( i = 1, \ldots, 6 \).

- Maximize the total NPV of the selected projects:
  \[ 141X_1 + 187X_2 + 121X_3 + 83X_4 + 265X_5 + 127X_6 \]
Solution to Lecture 4 Homework, Q.1.a.

Constraints

• Capital Constraints
  75X_1 + 90X_2 + 60X_3 + 30X_4 + 100X_5 + 50X_6 <= 250 \text{ year 1} \\
  25X_1 + 35X_2 + 15X_3 + 20X_4 + 25X_5 + 20X_6 <= 75 \text{ year 2} \\
  20X_1 + 0X_2 + 15X_3 + 10X_4 + 20X_5 + 10X_6 <= 50 \text{ year 3} \\
  15X_1 + 0X_2 + 15X_3 + 5X_4 + 20X_5 + 30X_6 <= 50 \text{ year 4} \\
  10X_1 + 30X_2 + 15X_3 + 5X_4 + 20X_5 + 40X_6 <= 50 \text{ year 5} \\

• Binary Constraints
  X_i <= 1, i = 1, 2, \ldots, 6 \\
  X_i >= 0, i = 1, 2, \ldots, 6 \\
  All X_i must be integers

• Solution Excel file available on website

Solution to Lecture 4 Homework, Q1.b.

Binary Variables & Logical Conditions

Binary variables are also useful in modelling a number of logical conditions.

– Of projects 1, 3 & 6, no more than one may be selected
  • X_1 + X_3 + X_6 <= 1

– Of projects 1, 3 & 6, exactly one must be selected
  • X_1 + X_3 + X_6 = 1

– Project 4 cannot be selected unless project 5 is also selected
  • X_4 - X_5 <= 0
Lecture 4 Homework, Q2.

2. In the Fixed Charge example we have seen how to model a logical expression of the form “if A then B” without using the IF-function. This is important since the solver does not interpret the IF function correctly. Instead we used a binary integer variable and a constraint to do this. (See website for Excel file of model given in Lecture 4.)

Change the model to allow for a discount in total fixed costs of £200 if two products are produced together and of £300 if all three products are produced. Don’t use the MIN or MAX functions because they take you out of the realm of linear problems and the solver may therefore have difficulties.

MAX: 48X_1 + 55X_2 + 50X_3
     \quad - 1000Y_1 - 800Y_2 - 900Y_3
Subject to:
  2X_1 + 3X_2 + 6X_3 \leq 600  \quad \text{machining}
  6X_1 + 3X_2 + 4X_3 \leq 300  \quad \text{grinding}
  5X_1 + 6X_2 + 2X_3 \leq 400  \quad \text{assembly}
  X_1 - 50Y_1 \leq 0
  X_2 - 67Y_2 \leq 0 \quad \text{linking}
  X_3 - 75Y_3 \leq 0
Y_i \text{ binary} \quad \text{binary}
X_i \geq 0 \quad i = 1, 2, 3 \quad \text{nonnegativity}
Solution to Lecture 4 Homework, Q2.

Adding Discounts to Model

• Add two more binary variables,
  – \( Y_4 = 1 \) if exactly two products are produced, 0 otherwise.
  – \( Y_5 = 1 \) if all three products are produced, 0 otherwise.

• Profit objective as before + 200Y_4 + 300Y_5

• New linking constraints
  - \( 2Y_4 \leq Y_1+Y_2+Y_3 \) or \(-Y_1-Y_2-Y_3+2Y_4\leq 0\)
    \( (Y_4 = 0 \) if fewer than 2 products are produced\)
  - \( Y_4 \leq 3 - (Y_1 + Y_2 + Y_3) \) or \( Y_1+Y_2+Y_3+Y_4 \leq 3\)
    \( (Y_4 = 0 \) if 3 products are produced\)
  - \( 3Y_5 \leq Y_1+Y_2+Y_3 \) or \(-Y_1-Y_2-Y_3+3Y_5\leq 0\)
    \( (Y_5 = 0 \) if fewer than 3 products are produced\)

Lecture 4 3E4 Homework. Q.3

3. Consider the Air Express problem in the next slide Model on paper and solve in Excel
(a) as an LP
(b) after adding integrality constraints. Re-Solve using your last solution as a starting point. Repeat this process …
What do you notice?
Should you be concerned?
Employee Scheduling Problem: Air-Express

<table>
<thead>
<tr>
<th>Day of Week</th>
<th>Workers Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sunday</td>
<td>18</td>
</tr>
<tr>
<td>Monday</td>
<td>27</td>
</tr>
<tr>
<td>Tuesday</td>
<td>22</td>
</tr>
<tr>
<td>Wednesday</td>
<td>26</td>
</tr>
<tr>
<td>Thursday</td>
<td>25</td>
</tr>
<tr>
<td>Friday</td>
<td>21</td>
</tr>
<tr>
<td>Saturday</td>
<td>19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shift</th>
<th>Days Off</th>
<th>Wage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Sun &amp; Mon</td>
<td>£680</td>
</tr>
<tr>
<td>2</td>
<td>Mon &amp; Tue</td>
<td>£705</td>
</tr>
<tr>
<td>3</td>
<td>Tue &amp; Wed</td>
<td>£705</td>
</tr>
<tr>
<td>4</td>
<td>Wed &amp; Thr</td>
<td>£705</td>
</tr>
<tr>
<td>5</td>
<td>Thr &amp; Fri</td>
<td>£705</td>
</tr>
<tr>
<td>6</td>
<td>Fri &amp; Sat</td>
<td>£680</td>
</tr>
<tr>
<td>7</td>
<td>Sat &amp; Sun</td>
<td>£655</td>
</tr>
</tbody>
</table>

Objective:
Allocate workers to (5 day) shifts in order to meet scheduling requirements at least cost

Solution to Lecture 4 Homework, Q3.

Air-Express:

Defining the Decision Variables

\[ X_1 = \text{the number of workers assigned to shift 1} \]
\[ X_2 = \text{the number of workers assigned to shift 2} \]
\[ X_3 = \text{the number of workers assigned to shift 3} \]
\[ X_4 = \text{the number of workers assigned to shift 4} \]
\[ X_5 = \text{the number of workers assigned to shift 5} \]
\[ X_6 = \text{the number of workers assigned to shift 6} \]
\[ X_7 = \text{the number of workers assigned to shift 7} \]
Solution to Lecture 4 Homework, Q3.

**Air-Express:**

*Defining the Objective Function*

Minimize the total wage expense.

MIN: $680X_1 + 705X_2 + 705X_3 + 705X_4 + 705X_5 + 680X_6 + 655X_7$

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Solution to Lecture 4 Homework, Q3.

**Air-Express:**

*Defining the Constraints*

- Workers required each day
  - $0X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 0X_7 \geq 18 \}$ Sunday
  - $0X_1 + 0X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 \geq 27 \}$ Monday
  - $1X_1 + 0X_2 + 0X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 \geq 22 \}$ Tuesday
  - $1X_1 + 1X_2 + 0X_3 + 0X_4 + 1X_5 + 1X_6 + 1X_7 \geq 26 \}$ Weds.
  - $1X_1 + 1X_2 + 1X_3 + 0X_4 + 0X_5 + 1X_6 + 1X_7 \geq 25 \}$ Thurs.
  - $1X_1 + 1X_2 + 1X_3 + 1X_4 + 0X_5 + 0X_6 + 1X_7 \geq 21 \}$ Friday
  - $1X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 0X_6 + 0X_7 \geq 19 \}$ Saturday

- Nonnegativity conditions
  - $X_i \geq 0 \text{ for all } i$
Solution to Lecture 4 Homework, Q3.

LP and ILP Solutions to Air Express Problem

• The LP solution was non-integral, i.e. recommended that the 7 shifts be composed of
  \[1.33, 4, 6.33, 0, 7.33, 0.33, 13.00\] workers
  with a wages cost of \(£22,103\).

• Let’s impose the extra constraint that the no. of workers on each shift is an integer.

• This yields a new solution of
  \[1, 5, 6, 1, 6, 0, 13\] workers
  with slightly higher cost \(£22,540\).

• Re-solving yields different solutions – there are many possible, and this is not a problem.

• See website for Excel solution file

Lecture 4 3E4 Homework, Q.4

4. Try the Cable Problem: maximise profit which is £5 per metre of cable produced, where there is a set up cost of £50 and a limit on production capacity of 20 metres.

(a) Use the Excel \textbf{IF} function to formulate:

\[
\begin{align*}
\text{min} & \quad (5X - 50 \text{ if } X > 0, \ 0 \text{ otherwise}) \\
\text{subject to} & \quad 0 \leq X \leq 20 \\
\text{starting from} & \quad X = 0.
\end{align*}
\]

(b) Re-do this as a fixed cost integer LP

(c) What is the difference?
Solution to Lecture 4 Homework, Q.4

4. Excel file for (a), (b) available on website.
   (a) Solver stops at X=0, i.e. suggests this as the optimal solution with profit 0.
   (b) Re-do this as a fixed cost integer LP – see excel file on website – generates answer X=20 with profit 50.
   (c) What is the difference? – The ILP solution is the true (“global”) optimal solution, you can’t get a better profit than 50. The solution in (a) is a “local” solution; excel couldn’t do better by taking guesses for X near 0.

Lecture 5 Homework, Q1

1.a Model the ACA’s shortest path problem see next two slides) to find the quickest route (minimizing travel time).

1.b Model the ACA’s shortest path problem to find the the most scenic route (maximizing the scenic rating points)

1.c Solve (using Excel) either of these shortest path problems.

For each question assume node 1 as origin and node 11 as destination.
The American Car Association

The ACA provides a service to its clients: you provide your origin and destination cities, then it determines the quickest way (shortest path) for you to drive there.
Solution to

Lecture 5 Homework Q1.

1.a min

\[ 2.5X_{12} + 3X_{13} + 1.7X_{23} + 2.5X_{24} + 1.7X_{35} + 2.8X_{36} + 2X_{46} + 1.5X_{47} + 2X_{46} + 5X_{39} + 3X_{68} + 4.7X_{69} + 1.5X_{78} + 2.3X_{7,10} + 2X_{89} + 1.1X_{8,10} + 3.3X_{9,11} + 2.7X_{10,11} \]

s.t.

\[-X_{12} - X_{12} = -1 \quad \text{(flow constraint for node 1)}\]
\[+X_{9,11} + X_{10,11} = 1 \quad \text{(flow constraint for node 11)}\]
\[X_{12} - X_{23} - X_{24} = 0 \quad \text{(flow constraint for node 2)}\]
\[X_{13} + X_{23} - X_{35} - X_{36} = 0 \quad \text{(flow constr. for node 3)}\]
\[X_{24} - X_{46} - X_{47} = 0 \quad \text{(flow constr. for node 4)}\]

1.b Same constraints as above. Just change objective function.

1.c See the file on the website.

Lecture 5 3E4 Homework Q2.

2 The Equipment Replacement Problem

The problem of determining when to replace equipment is a common business problem. It can also be modeled as a shortest path problem.

2.a Model the following Compu-Train replacement problem as a pair of shortest path problems, comparing the two solutions.
The Compu-Train Company

• Compu-Train provides hands-on software training.
• Computers must be replaced at least every two years.
• Two lease contracts are being considered:
  – Each required $62,000 initially
  – Contract 1:
    • Prices increase 6% per year
    • 60% trade-in for 1 year old equipment
    • 15% trade-in for 2 year old equipment
  – Contract 2:
    • Prices increase 2% per year
    • 30% trade-in for 1 year old equipment
    • 10% trade-in for 2 year old equipment
• Want to determine which contract would allow to minimize the remaining leasing cost over the next five years and when, under the selected contract, the equipment should be replaced.

Solution to
Lecture 5 Homework Q2.

Network for Contract 1

Cost of trading after 1 year: 1.06*$62,000 - 0.6*$62,000 = $28,520
Cost of trading after 2 years: 1.06^2*$62,000 - 0.15*$62,000 = $60,363
etc, etc....
See also Excel file on the website.
3.a
Given the network in the next slide, determine the maximum flow that is possible to send from $s$ to $t$ using the Ford-Fulkerson algorithm. Arc capacities are showed in the picture along each arc.

3.b
Check the optimality of the solution using the max flow – min cut theorem.

3.c
Solve the problem using Excel.
Solution to Lecture 5 Homework 3a.

- L:=\{s\}; Extract s; L:=\emptyset;
- Scan s:
  - P(1)=s,F(1)=2,
  - P(2)=s,F(2)=2;
  - L:=\{1,2\};
- Extract 1; L:=\{2\};
- Scan 1:
  - P(4)=1,F(4)=\min\{2,7\}=2,
  - L:=\{2,4\};
- Extract 4; L:=\{2\};
- Scan 4:
  - P(t)=4,F(t)=\min\{2,1\}=1;
- t is labeled, increase of F(t)=1 the flow along the augmenting path
  \text{t} \leftarrow P(t)=4 \leftarrow P(4)=1 \leftarrow P(1)=s.

\[\begin{array}{c}
\text{s} \\
\text{2} & \text{3} \\
\text{1} & \text{4} & \text{t}
\end{array}\]

\[\begin{array}{c}
[0,2] & [0,2] \\
[0,5] & [0,2] \\
[0,7] & [0,7] & [0,3] \end{array}\]

\[\begin{array}{c}
[0,2] \\
[0,1] \\
[0,3] \\
[0,2]
\end{array}\]

Solution to Lecture 5 Homework 3a.

- L:=\{s\}; Extract s; L:=\emptyset;
- Scan s:
  - P(1)=s,F(1)=1,
  - P(2)=s,F(2)=2;
  - L:=\{1,2\};
- Extract 2; L:=\{1\};
- Scan 2:
  - P(3)=2,F(3)=\min\{2,2\}=2,
  - L:=\{1,3\};
- Extract 3; L:=\{1\};
- Scan 3:
  - P(4)=3,F(4)=\min\{2,2\}=2;
  - P(t)=3,F(t)=\min\{2,3\}=2;
- t is labeled, increase of F(t)=2 the flow along the augmenting path
  \text{t} \leftarrow P(t)=3 \leftarrow P(3)=2 \leftarrow P(2)=s.

\[\begin{array}{c}
\text{s} \\
\text{2} & \text{3} \\
\text{1} & \text{4} & \text{t}
\end{array}\]

\[\begin{array}{c}
[1,2] & [0,2] \\
[0,5] & [0,2] \\
[1,7] & [0,7] & [0,3] \end{array}\]

\[\begin{array}{c}
[1,2] \\
[1,1] \\
[1,7] \\
[0,2]
\end{array}\]
Solution to Lecture 5 Homework 3a.

- $L := \{s\};$ Extract $s; L := \emptyset$;
- Scan $s$:
  - $P(1) = s, F(1) = 1,$
  - $L := \{1\}$;
- Extract 1; $L := \emptyset$;
- Scan 1:
  - $P(4) = 1, F(4) = \min\{1, 6\} = 1,$
  - $L := \{4\}$;
- Extract 4; $L := \emptyset$;
- Scan 4:
  - No more labeling allowed!
- List empty and $t$ unlabeled, then terminate.
- Labeled nodes are $s, 1, 4$. Unlabeled nodes are 2, 3, $t$.

\[ \begin{align*}
\text{W} &= \{2, 3, t\} \\
\text{V} &= \{s, 1, 4\} \\
C^*(V, W) &= c_{s2} + c_{4t} = 3 \\
f^* &= x_s + x_t = 3
\end{align*} \]

3.b See the Excel file on the website.

Solution to Lecture 5 Homework Q3
Lecture 6 Homework Q1.

1. (From 2001 exam) A tract of land of 1000 acres, owned by a local government council, is used as a bird sanctuary, for sheep grazing, and for recreation (walking). The council is reviewing the uses of this land and has designed indices to show the benefits of its use. All 1000 acres contribute to bird life. The bird index is 2 for each acre not used for grazing or recreation, 1 for each acre used for either grazing or recreation but not both, and 0 for each acre jointly used for grazing and recreation. The grazing index is 2 for each acre used for grazing but not recreation, 1 for each acre used for both grazing and recreation, and 0 otherwise. The recreation index is 2 for each acre used for recreation, 0 otherwise. After consulting community groups and other stakeholders, the council sets the following goals: the bird, grazing and recreation indices should exceed 3000, 1000 and 1000 respectively.

Formulate (mathematically) a goal program for this problem using percentage deviations.

Solution to Lecture 6 Homework Q1.

1. Define the variables:
   \( x_1 \) = no. acres of land used for birds only
   \( x_2 \) = no. acres of land used for (birds and) grazing but not recreation
   \( x_3 \) = no. acres of land used for (birds and) recreation but not grazing
   \( x_4 \) = no. acres of land used for (birds and) grazing and recreation

   Let \( w_1, w_2, w_3 \) be nonnegative weights (given), and \( s_1, s_2, s_3 \) be nonnegative deviational variables corresponding to each goal.

   The percentage deviations goal program is
   \[
   \text{Min } w_1 s_1 / 3000 + w_2 s_2 / 1000 + w_3 s_3 / 1000
   \]

   or in percentage
   \[
   \text{Min } w_1 s_1 / 30 + w_2 s_2 / 10 + w_3 s_3 / 10
   \]
Solution to

Lecture 6 Homework Q1.

Subject to the constraints:
\[ 2x_1 + x_2 + x_3 + s_1 \geq 3000 \text{ (birds)} \]
\[ 2x_2 + x_4 + s_2 \geq 1000 \text{ (grazing)} \]
\[ 2x_3 + 2x_4 + s_3 \geq 1000 \text{ (recreation)} \]
\[ x_1 + x_2 + x_3 + x_4 = 1000 \]
\[ x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0 \]

Solution to

Lecture 6 Homework Q2.

2. Formulate and solve, in Excel, the minimax goal program associated with the bi-objective 3 Stock Investment Example.
   • use one of the spreadsheets on website as a starting point
   • you will first have to define the targets of your goals with Excel.
   • make the weights data cells in your spreadsheet

SOLUTION:
See Excel file markowitz–C.xls on website
3. Consider a bi-objective LP, with goals Max u and Max v which are characterised by the feasible set
\[0 \leq u \leq 3, \quad 0 \leq v \leq 3, \quad u + v \leq 4.\]

3.a Sketch the feasible region and show the Pareto optimal set (efficient frontier)
The efficient frontier is the line segment joining the points (3,1) and (1,3).

3.b. Give nonnegative weights w₁, w₂ such that the problem
\[\text{Max } w₁u + w₂v \text{ subject to above constraints}\]
has a non-efficient solution. What solution is this?
E.g., (w₁,w₂)=(1,0) yields line segment from (3,0) to (3,1) as LP solution set. But only (3,1) is Pareto optimal (efficient).