PART IB MECHANICS

SECTION 2 of 3

2. RIGID BODY DYNAMICS PART 1: INERTIA FORCES AND ENERGY

2.1 Centre of mass, moments of inertia

2.2 d'Alembert force for a particle

2.3 d'Alembert force and torque for a rigid body in plane motion

2.4 Kinetic energy of a rigid body in plane motion

2.5 Conservation of energy for conservative systems

2.6 Inertia forces in plane mechanisms

2.7 Method of virtual power

2.8 Inertia stress and bending

2.9 Balancing simple rotors

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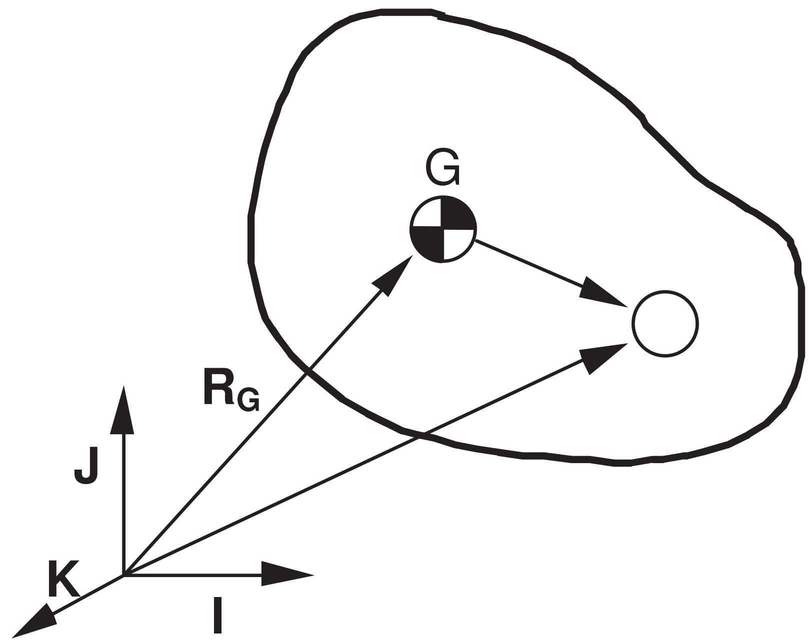
RIGID BODY DYNAMICS I: INERTIA FORCES, ENERGY

2.1 CENTRE OF MASS, MOMENTS OF INERTIA

CENTRE OF MASS OF A RIGID BODY



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The centre of mass is defined as the point for which:

(2.1)

(2.2)

(2.3)

$$\mathbf{R_G} = \frac{\sum m_i R_i}{M} = \frac{1}{M}\int \mathbf{R}\,dm \quad (2.4)$$

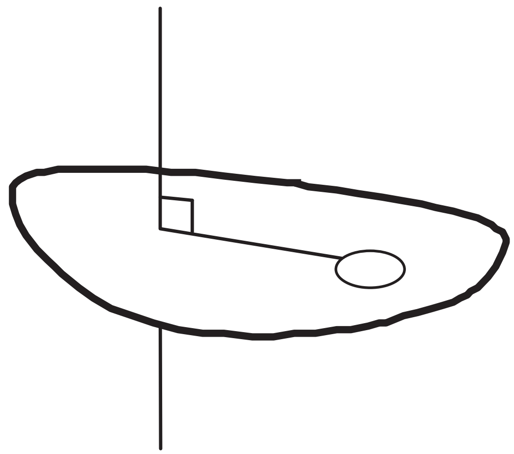
In cartesian coordinates:

(2.5)

So equating components:

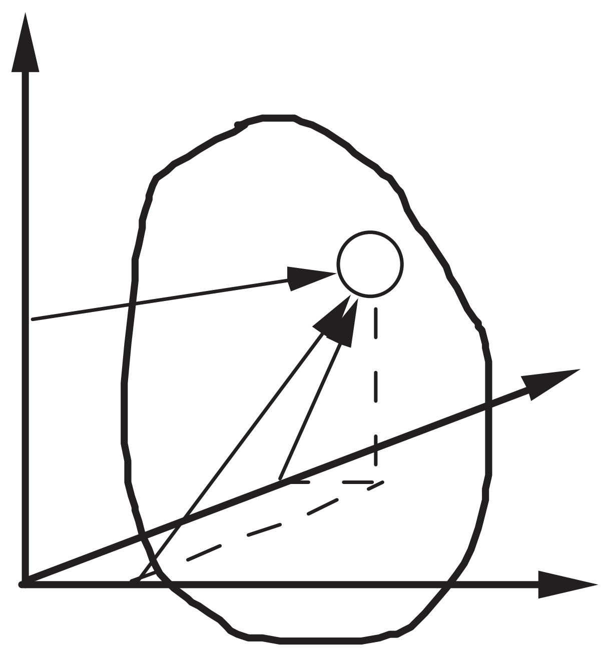
$$\bar{x} = \frac{1}{M}\int x\,dm, \quad \bar{y} = \frac{1}{M}\int y\,dm, \quad \bar{z} = \frac{1}{M}\int z\,dm \quad (2.6)$$

MOMENTS OF INERTIA OF SOLID BODIES



$$I_{zz} = \int r^2 dm \tag{2.7}$$

The moment of inertia is a measure of resistance to rotational acceleration, just as mass is a measure of resistance to linear acceleration.



Definitions – Moments of Inertia:

$$I_{xx} =$$

$$I_{yy} = \qquad (2.8)$$

$$I_{zz} =$$

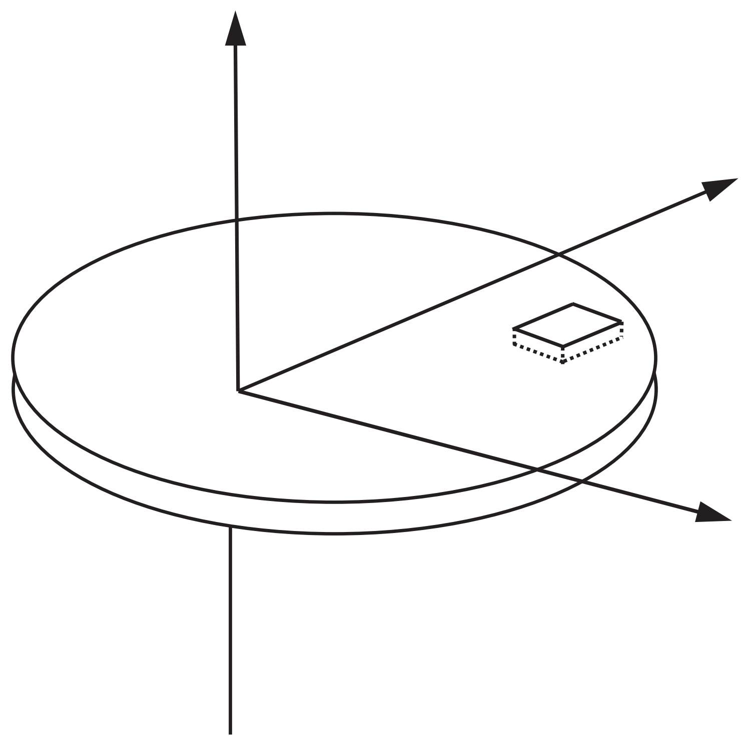
Definitions – Products of Inertia:

$$I_{xy} = \int xy\, dm, \quad I_{xz} = \int xz\, dm, \quad I_{yz} = \int yz\, dm \qquad (2.9)$$

The products of Inertia can be positive or negative. Note that products of inertia are included here for completeness. They are not part of the syllabus. See Part IIA Module 3C5 for proper treatment.

MOMENTS OF INERTIA OF LAMINAS

Suppose the body is thin in the z-direction



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(2.10)

(2.11)

$$I_{zz} = \int \left(x^2 + y^2 \right) dm = I_{xx} + I_{yy} \qquad (2.12)$$

Equation (2.12) is called the perpendicular axis theorem.

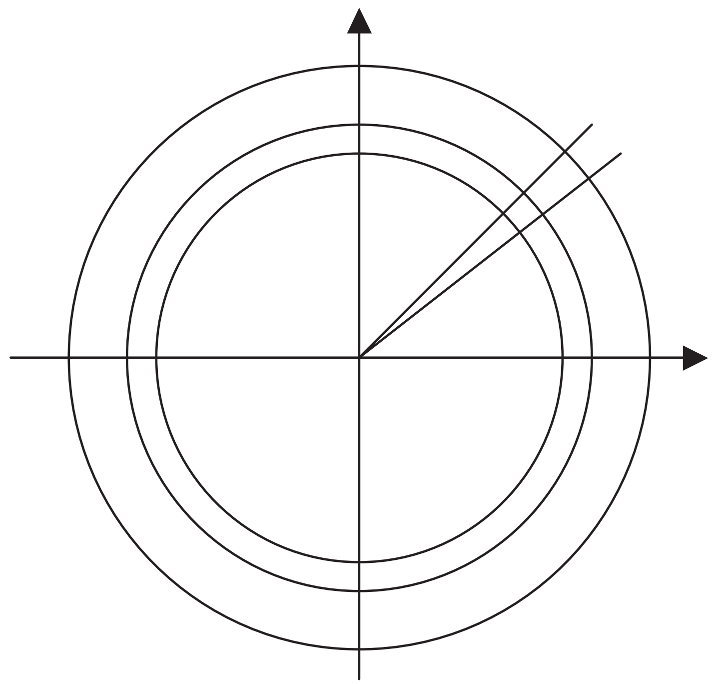
It applies to thin sheets (laminas) only.

RADIUS OF GYRATION

k is independent of mass – only depends on shape

The radii of gyration of various shapes are given in the Mechanics Data Book.

Example 2.1 ***Calculate the diametral moment of inertia of a thin circular disc of radius R and mass M***



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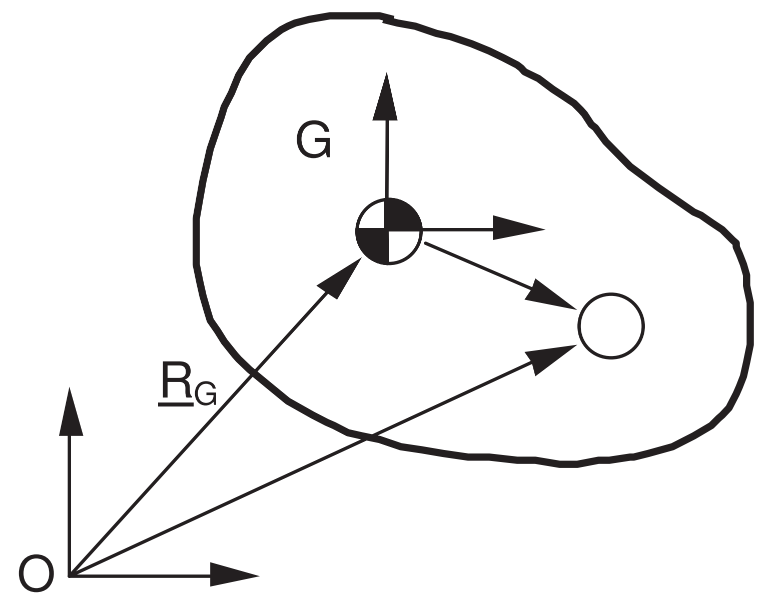
PARALLEL AXIS THEOREM



So $$\boxed{I_O = I_G + |\mathbf{R_G}|^2 M} \quad (2.14)$$

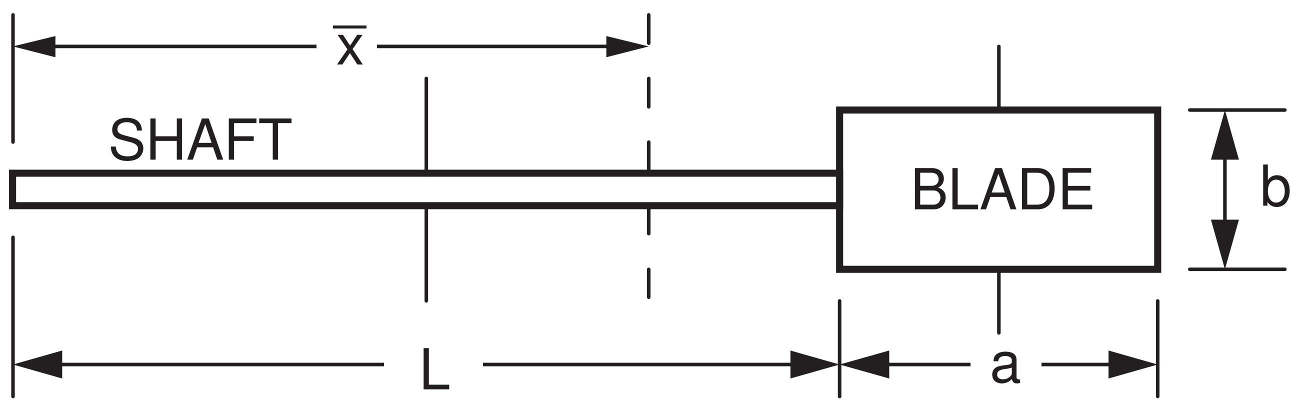
It is easy to show that this also applies for 3D rigid bodies, see 3C5 next year.

COMPOSITE BODIES

The moment of inertia of a composite body about an axis is the sum of the moments of inertia of the individual components about the same axis (via the parallel axis theorem). Moments of inertia of complex bodies can thus be calculated using data for simpler ones.

Note that a hole in a body can be considered to be an element of negative mass.

Example 2.2 ***Calculate the moment of inertia of an oar about a vertical axis through its end***



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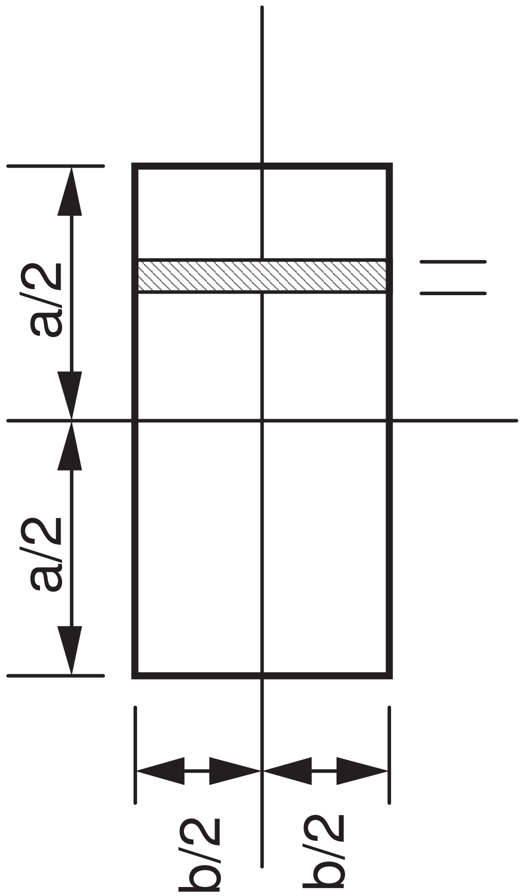
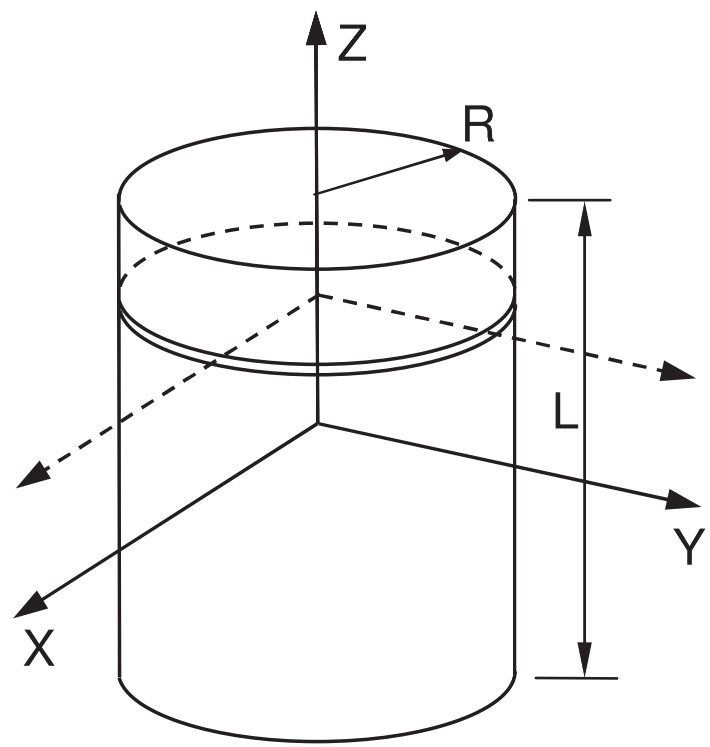

a/2
a/2
b/2
b/2

Example 2.3 Calculate the Moments of Inertia of a cylinder of Length L, Radius R, Mass M



From Ex 2.1, for a disc of mass *dm*:

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Now try Q1 on Examples sheet 2

I.1 D'ALEMBERT FORCE FOR A PARTICLE

For a particle, Newton II is $\sum \mathbf{F} = m\mathbf{a}$

$$ \tag{2.15}$$

From (2.15), inertia forces & applied forces form a system of "static equilibrium". We can therefore use the methods of statics, i.e. resolving, taking moments, Virtual Work, etc.

This looks trivial, but is very helpful, since dynamics becomes statics!

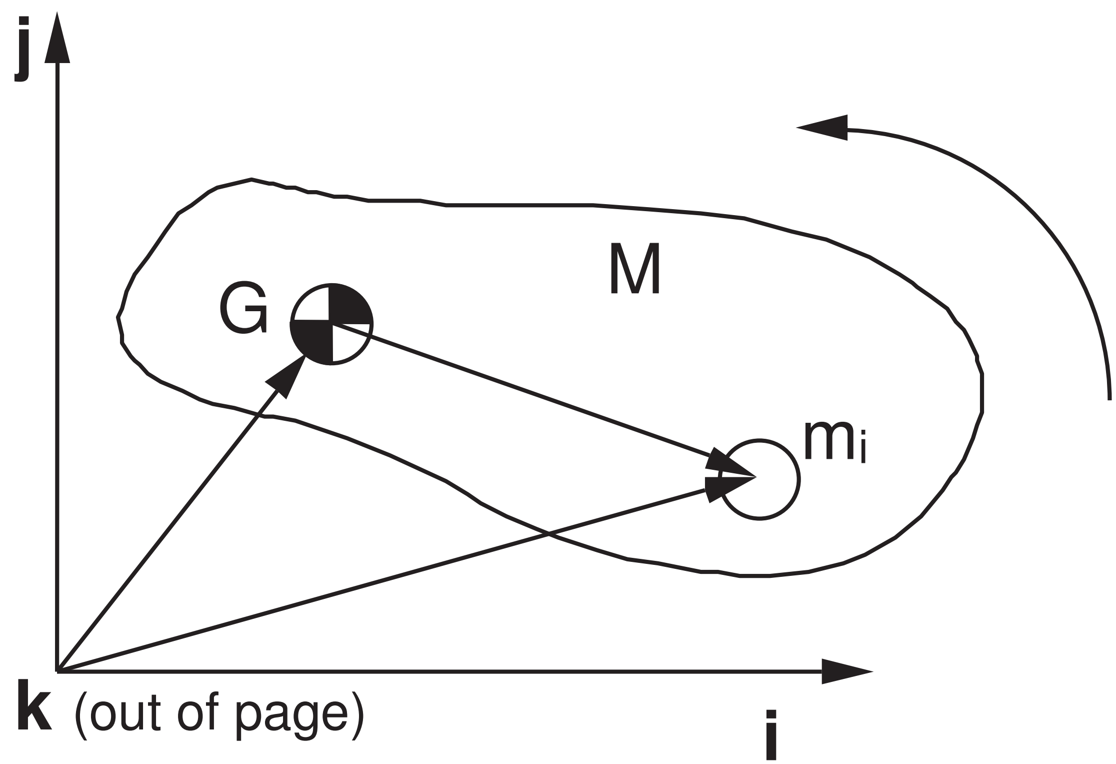
Example

Newton II:

D’Alembert:

I.2 D'ALEMBERT FORCE AND TORQUE FOR A RIGID BODY IN PLANE MOTION

What force / moment are needed to accelerate the body shown?



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Kinematics:

(1.4)

(1.8)

The total d'Alembert force on the body is:

So force equilibrium for the body is (equation 2.15)

(2.16)

This is exactly as for a particle with mass M, concentrated at G.

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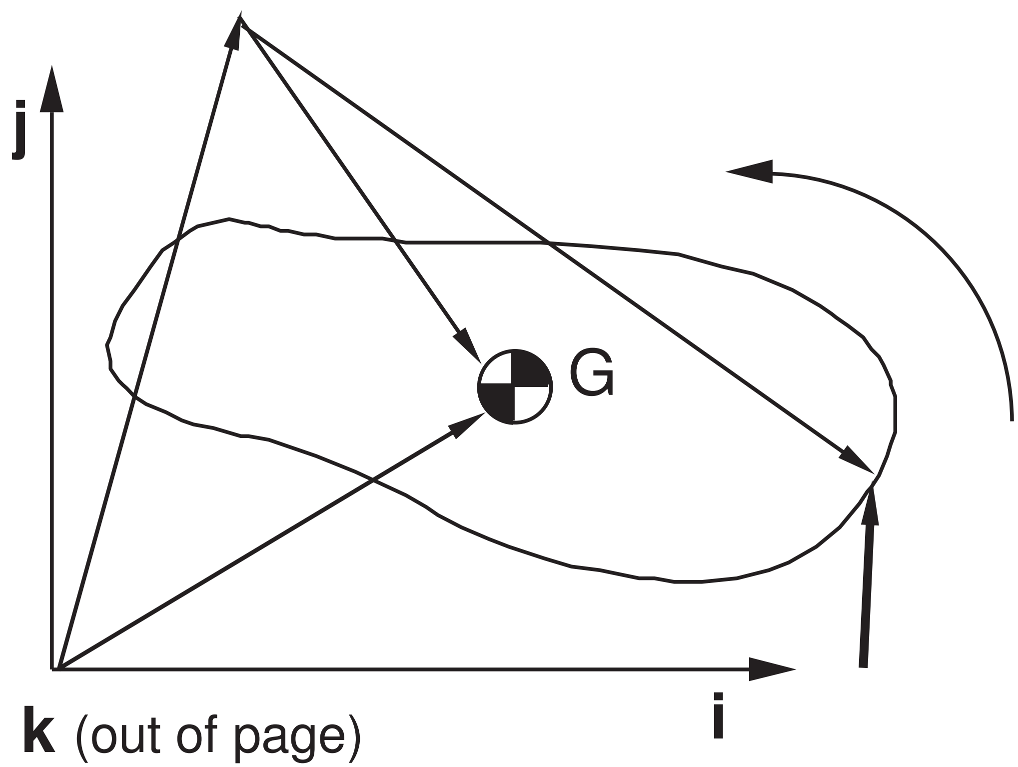
The total moment of the d'Alembert force about G is:

So moment equilibrium about G is:

$$\qquad (2.19)$$

Motion of a body with applied forces



Similarly, moment equilibrium about any other point O_2 is (see figure above):

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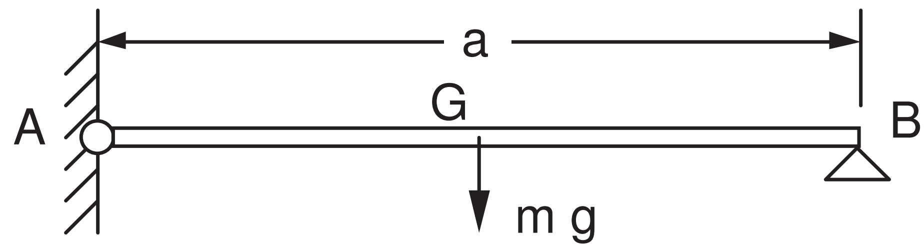
(2.20)

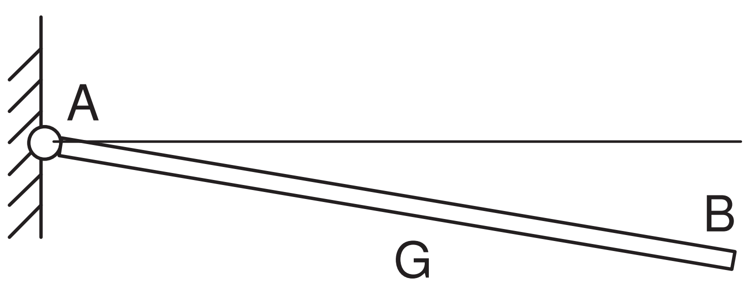
The problem can be treated exactly as static equilibrium of the real applied forces and moments and the *apparent* “Inertia Forces” and “Inertia Torques”

Example 2.4 Uniform bar falling from rest under gravity

What is the angular acceleration and reaction at A immediately after the removal of the support at B? What is the subsequent velocity?



Kinematics



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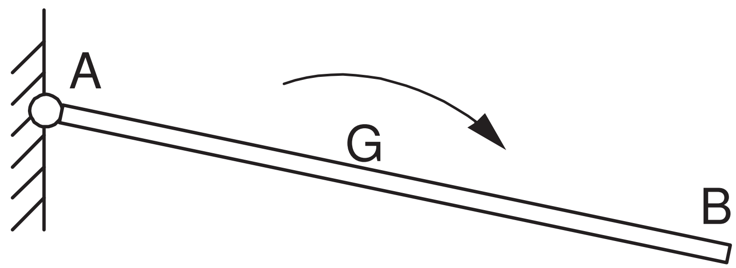
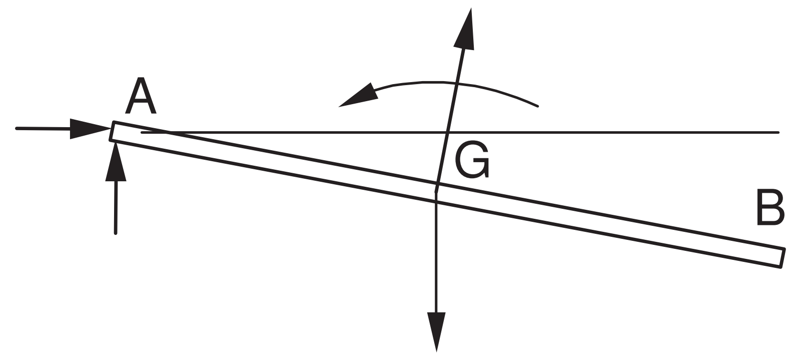


Accelerations



Forces

Taking moments about A (to eliminate H and V)

(A)

Resolving vertically

(B)

(C)

Note — We could have used the parallel axis theorem to find I_A and hence the angular acceleration, but this would have led to a mistake with the $ma\ddot{\theta}$ term when calculating the reaction V. Therefore never use the parallel axis theorem with d’Alembert (except to calculate I_G)

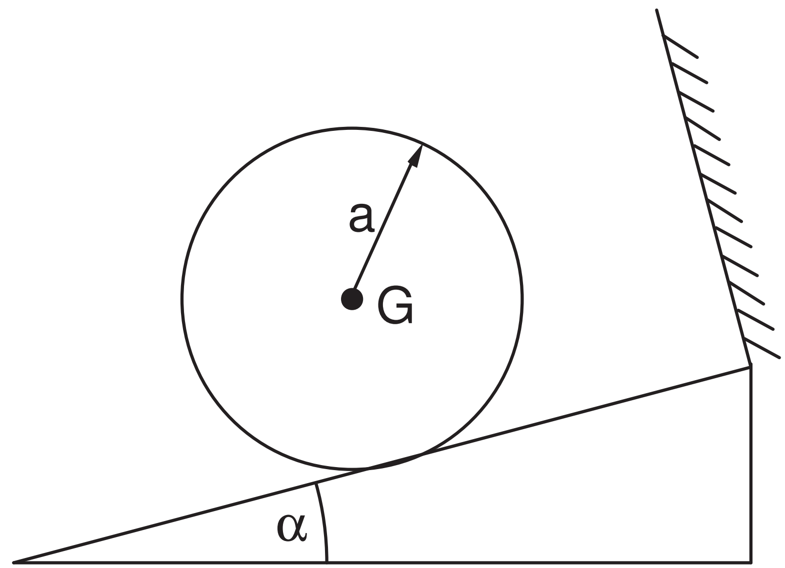
Velocities

Initial Reaction

Therefore, on release, V immediately **halves**.

Example 2.5 Sphere released on a rough inclined plane

What coefficient of friction μ is needed for no initial slip between sphere and plane?



Kinematics

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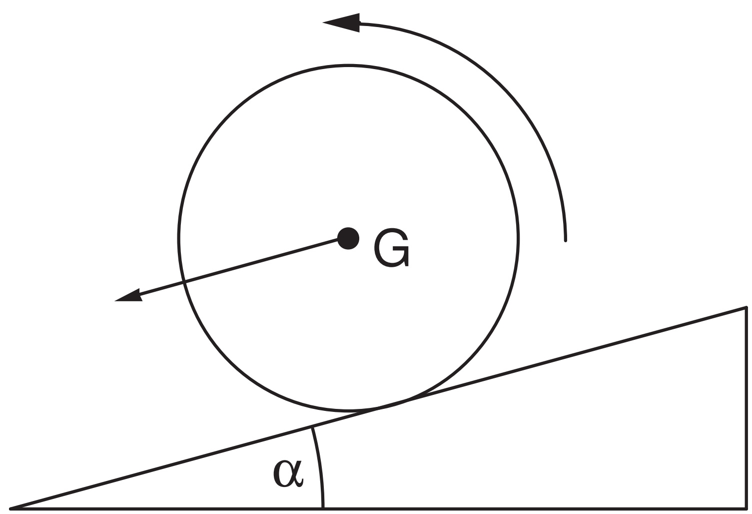
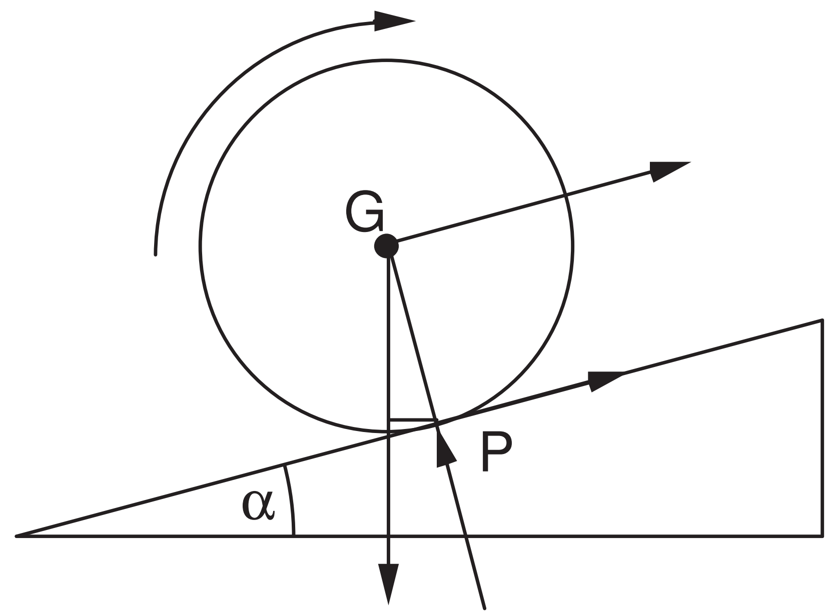


Accelerations



Forces

d'Alembert

Moments about P:

(A)

Resolve along slope:

(B)

Resolve perp to slope:

(C)

Initial Accelerations

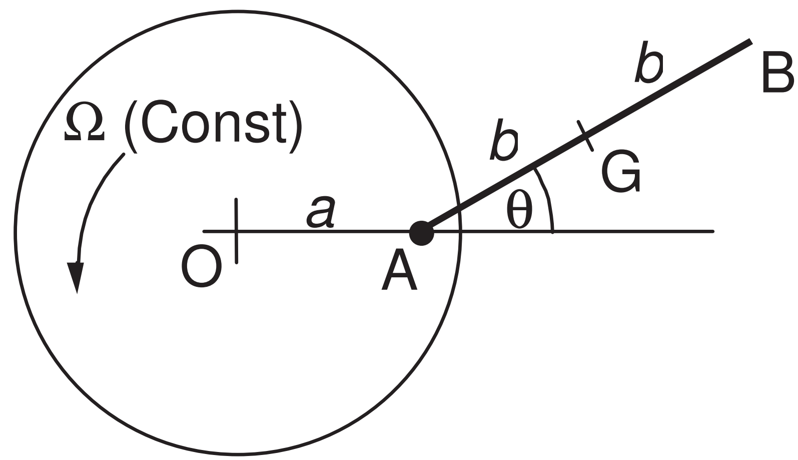
.---.

(D)

No slip condition

Example 2.6 Vibrating turbine blade

A turbine blade, considered to be a uniform bar of mass m and length $2b$, is hinged to the turbine disc at radius a. What is the frequency of small oscillations of the blade. Neglect gravity.



Kinematics

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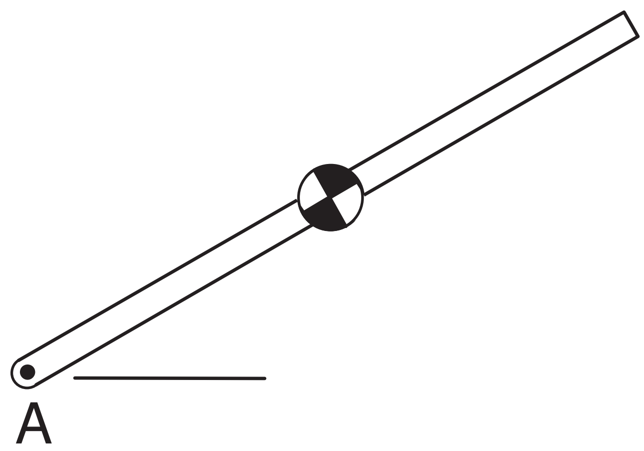
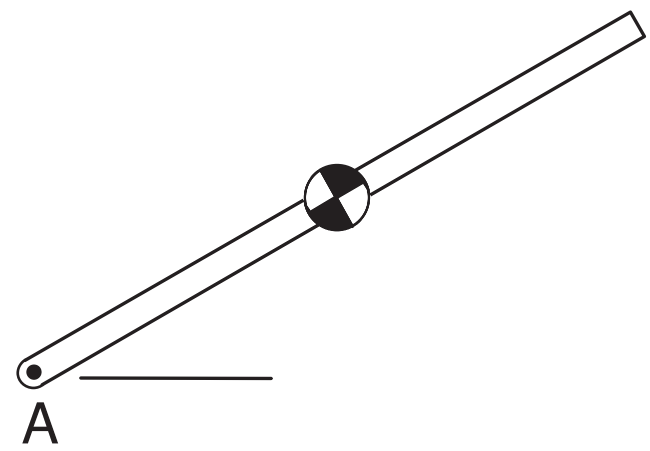


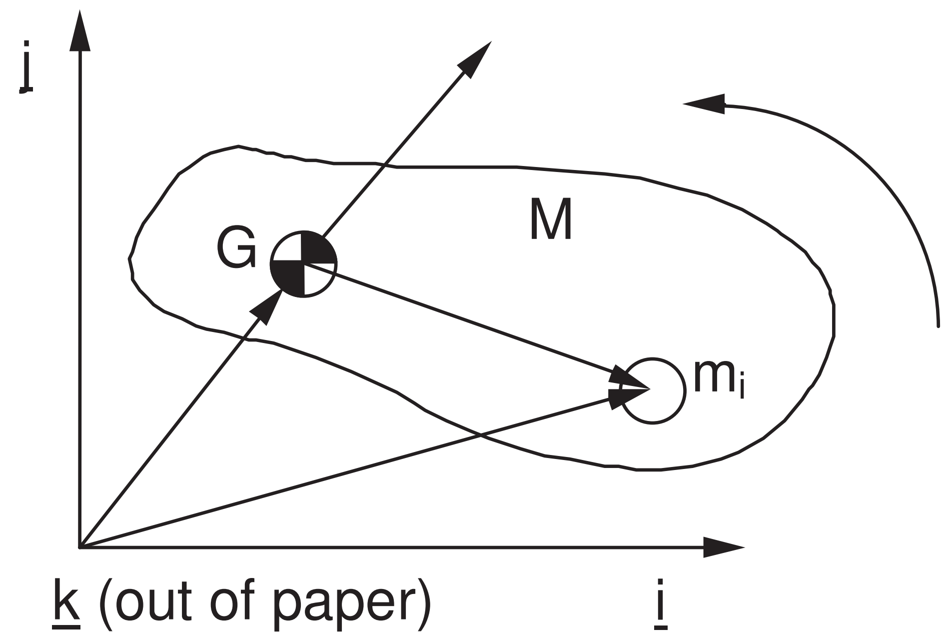
Accelerations



Forces

Now try questions 2 and 3 on Examples paper 2

2.4 Kinetic energy of a rigid body in plane motion



The K.E. of elemental mass m_i is:

The K.E. for the whole body is:

(2.21)

(2.22)

2.5 Conservation of energy for conservative systems

For a conservative system:

(2.23)

Where, for rigid bodies:

.---.

Notes:

1 Equation (2.23) is a scalar equation, which can replace one of the three scalar equations of motion for a 2-D system.

2 Unknown forces which do no work are automatically eliminated from the equation. If they have to be found, further calculation is needed.

3 Systems must be conservative - there must be no forces which dissipate energy, or which do work on the system.

4 If a system has only one degree of freedom, equation (2.23) can be used to find its motion.

Example 2.7 Example 2.4 by Energy

Kinetic Energy:

(A)

Potential Energy:

(B)

Conservation

(C)

(D)

Accelerations

Notes:

d’Alembert and the Energy method give the same equation of motion, but:

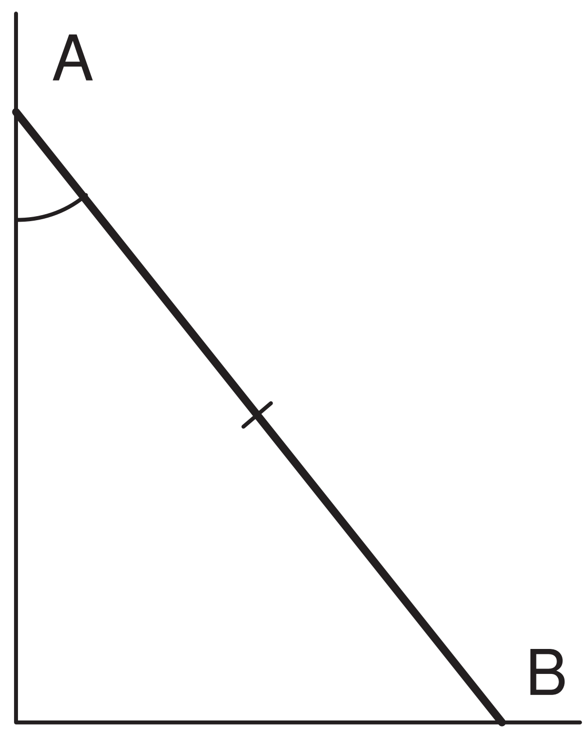
(i) d’Alembert gives equations for accelerations ➔ it is necessary to integrate these w.r.t time, to get velocities and displacements

(ii) Energy gives equations for velocities and displacements ➔ these must be differentiated to get accelerations

(iii) Forces can only be obtained from d’Alembert

Example 2.8 Ladder sliding down a wall

A uniform ladder of mass m is held against a smooth wall while standing on a smooth floor. The base of the ladder is moved slightly away from the wall and released. Determine $\dot{\theta}$ as a function of θ during the subsequent motion What angle does the ladder make with the vertical when contact with the wall ceases?



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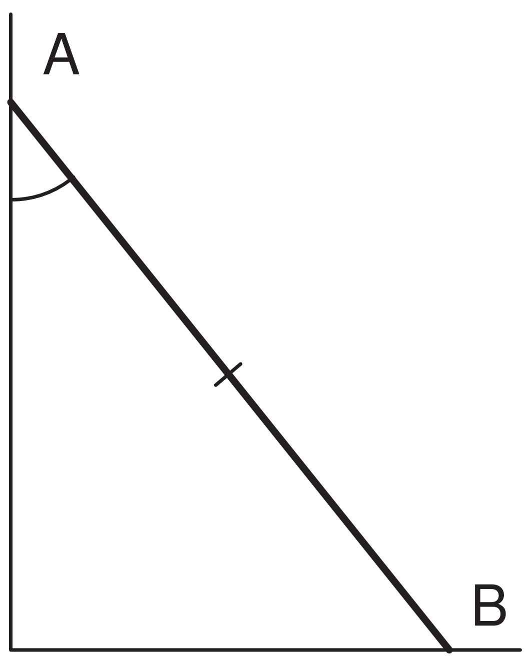
Kinematics

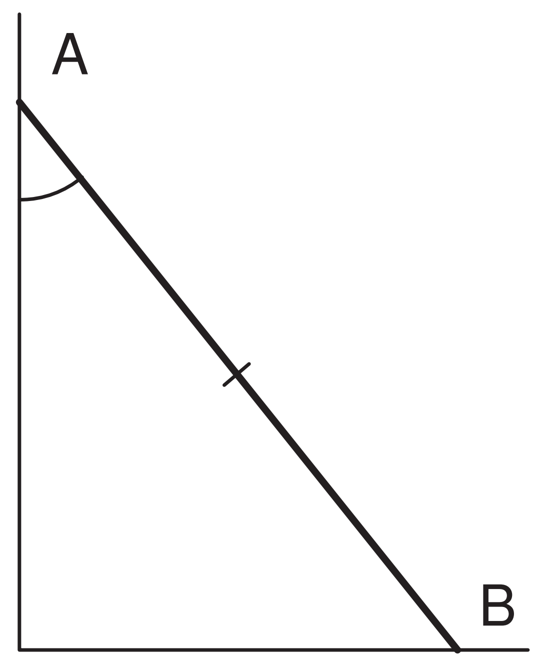
(A)

(B)

(C)



Accelerations



Forces

To find the equation of motion, use either:

(i) Sum moments about P – Eliminates V and H simultaneously

(ii) Energy – Neither V nor H do any work..

Kinetic Energy:

Potential Energy

.---.

(D)

(E)

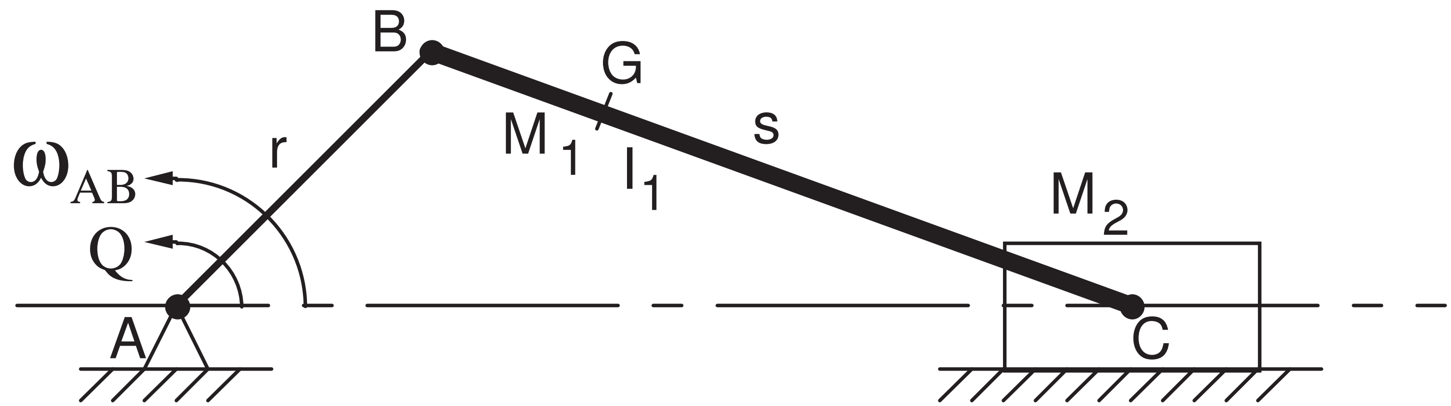
Loss of contact at A when $H = 0$. Use d'Alembert:

(F)

Now try questions 4-6 on Examples Paper 2

Example 2.9 Reciprocating engine

Determine the torque Q needed for constant crank angular velocity ω_{AB}..
Assume $m_{AB} = 0$ (all links are horizontal – no gravity)

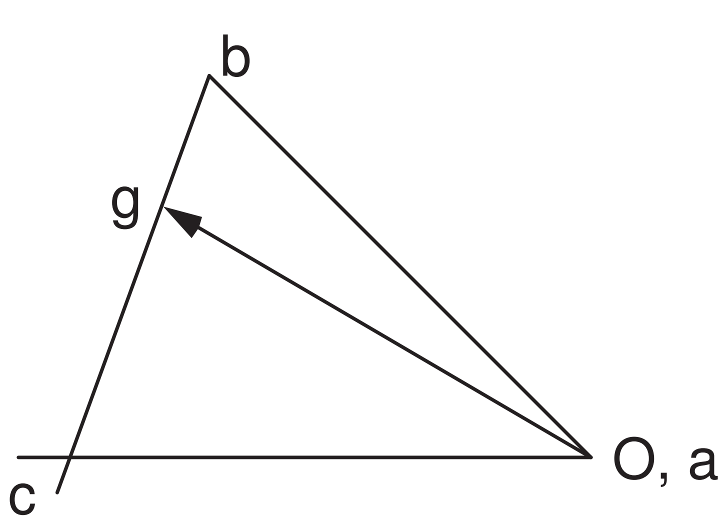


(A)

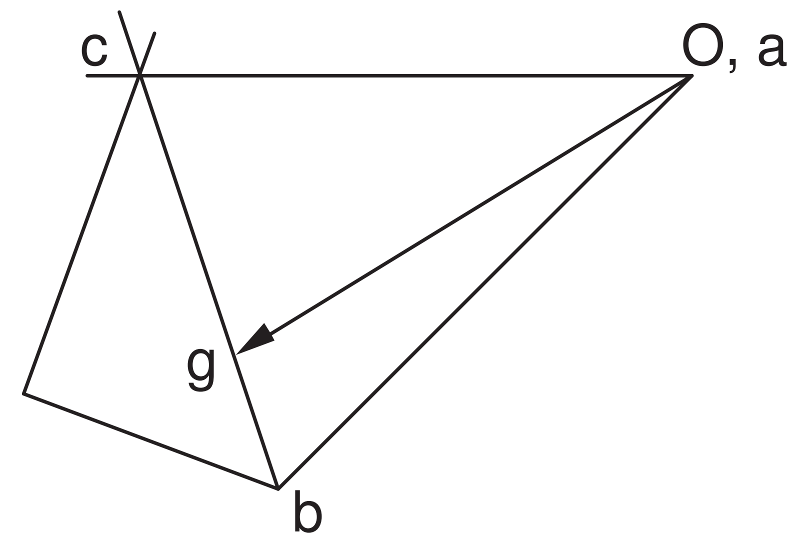
(B)

(C)

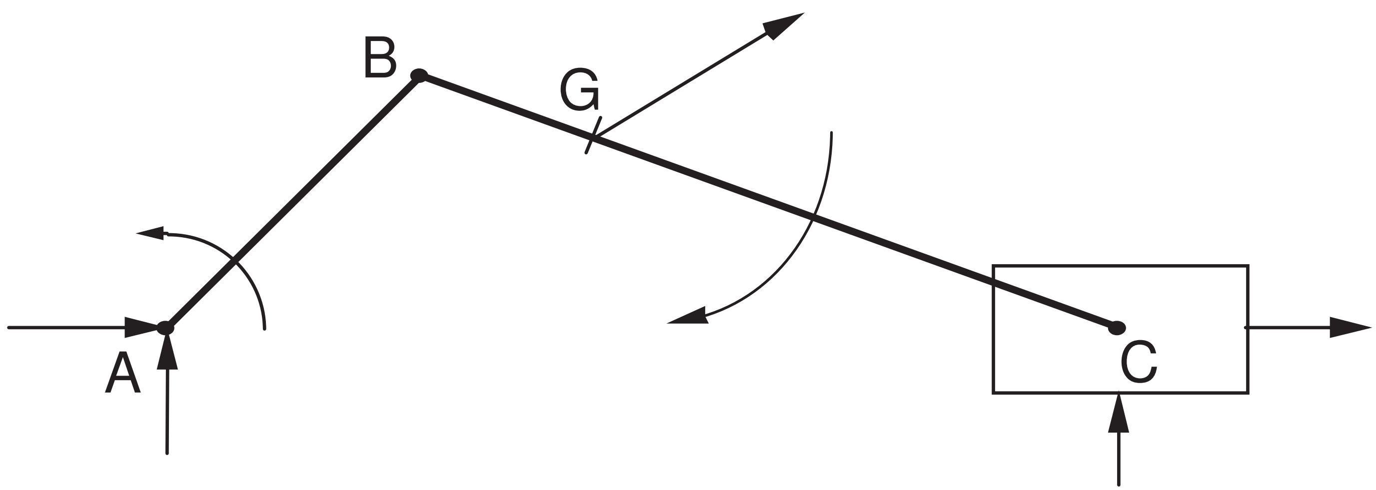
.---.



Velocity Diagram



Acceleration Diagram



Forces

Solution Method 1 - Statics

4 unknowns: Q, R, V, H (but we only want Q)

Can take moments / resolve for whole mechanism, or for individual members
(but remember internal forces in the joints)

Solution Method 2 - Virtual Work

.---.

2.7 METHOD OF VIRTUAL POWER

The Virtual Work equation (developed in Part I Structural Mechanics) is:

If the bodies are rigid, all the terms on the R.H.S. become zero. Adding a term for applied moments to the L.H.S. gives:

(2.24)

Notes:

(i) For convenience, use the real displacements of the mechanism in a small time δt

(2.25)

(ii) F_i's include the inertia forces $(-M\ddot{R}_G)$

(iii) M_j's include the inertia torques $(-I_G\ddot{\theta})$

(iv) Internal forces occur in equal and opposite pairs, therefore do no work:

Equations (2.24) and (2.25) therefore give:

(2.26)

(2.27)

.---.

Applying this to Example 2.9 gives: (Note V, H and R do no work)

Note: The 'initial motion' problem

If the mechanism is instantaneously at rest, the virtual power equation above cannot be applied directly since all the $\underline{v}$'s and $\underline{\omega}$'s are zero. It is perfectly legal to use *any* compatible set of velocities together with the true accelerations in (2.26). (See Part IA Structures)

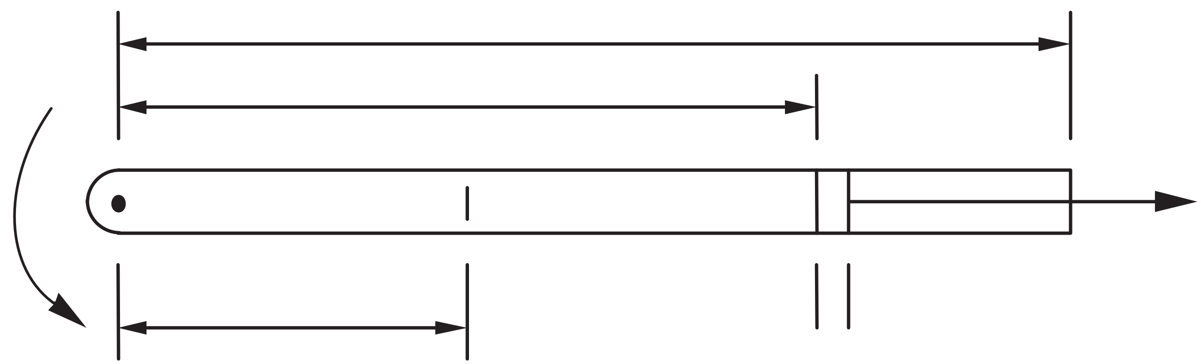
Now try questions 7-8 on Examples Paper 2

2.8 INERTIA STRESS AND BENDING

We now know what external forces and moments must be applied to a body to give it a particular motion. What internal stresses and bending moments are set up?

Example 2.10 Rotating uniform bar

Density ρ, c/s area A



Stress on a typical cross section at radius z is given by:

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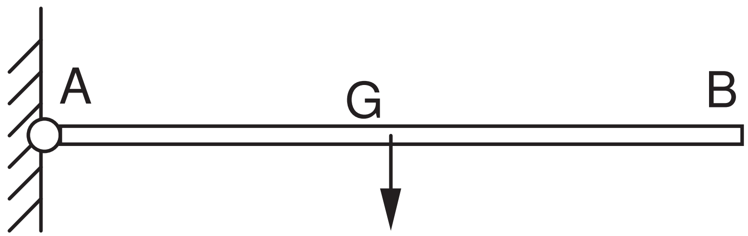
Plot of longitudinal stress distribution

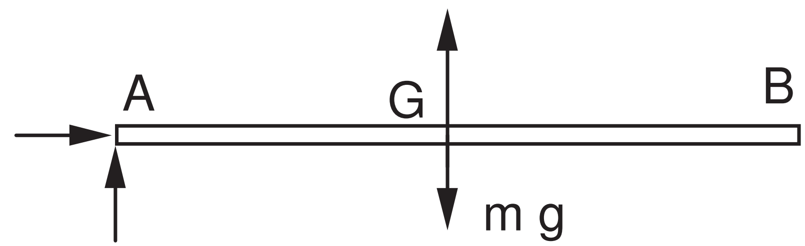
Revision of Shear Force & Bending Moment Sign Conventions

Example 2.11 *Shear and bending in a falling beam*

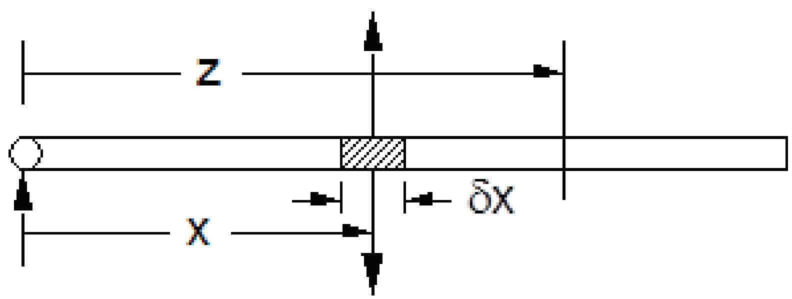
What is the distribution of shear force and bending moment in the beam in Example 2.4, immediately after removal of the support at B?

1 Overall considerations





2 Internal considerations



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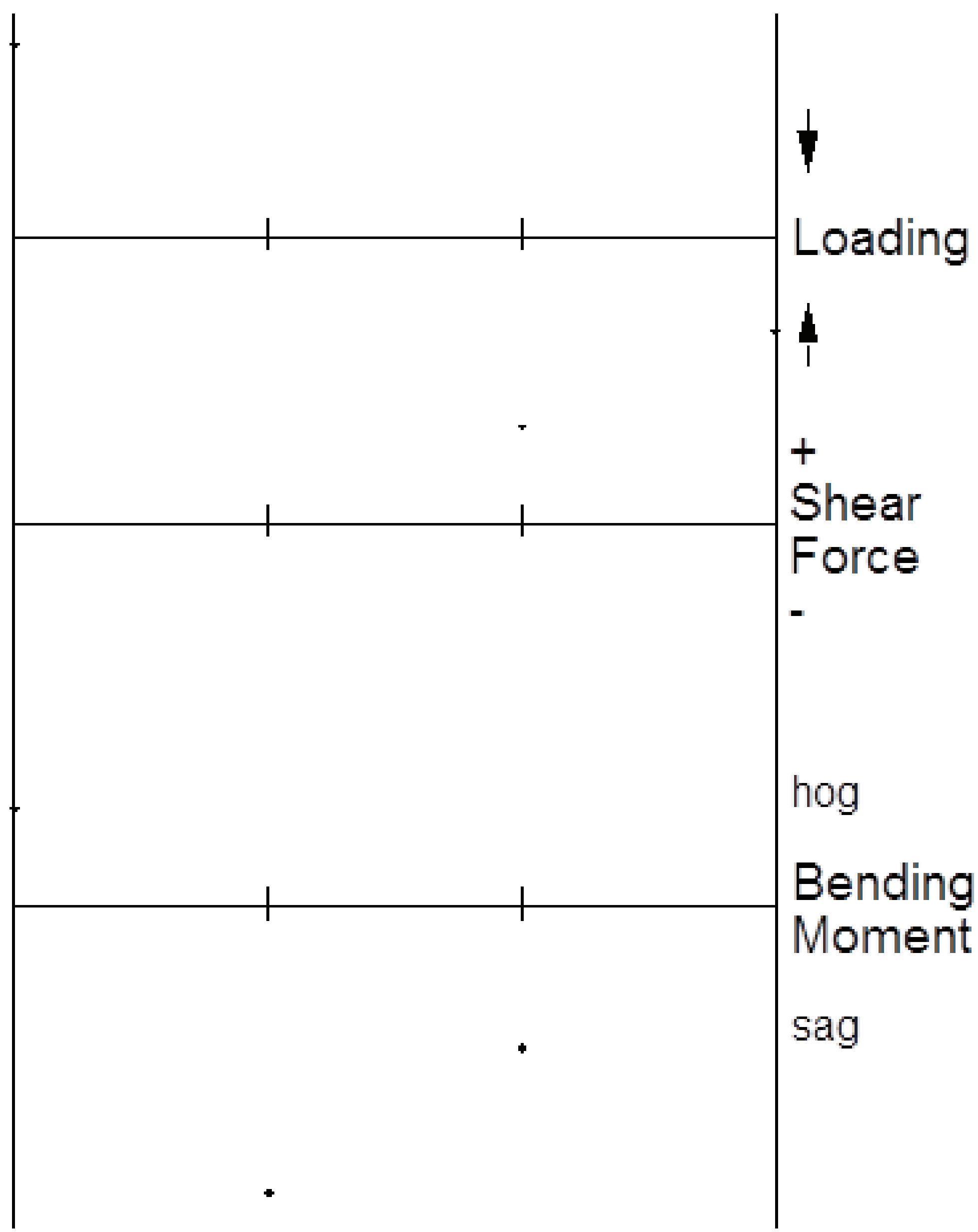
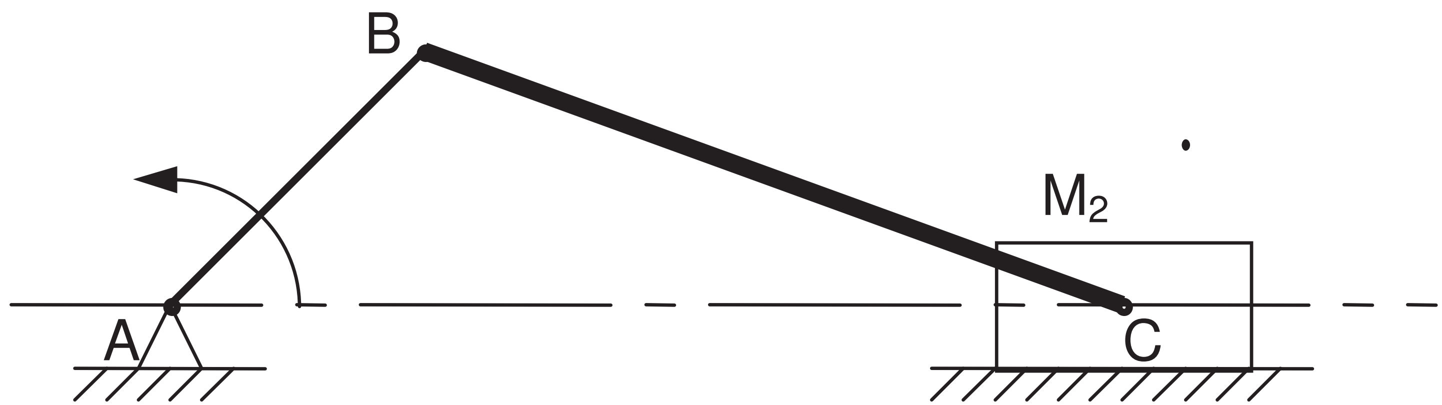

Loading
+
Shear
Force
-
hog
Bending
Moment
sag

Example 2.12: Falling Chimney Stack

The calculation is exactly the same as in Ex 2.11.

- The maximum bending moment is in the same place (1/3 of the height)
- Bricks have no tensile strength. Therefore the chimney breaks when the tension due to the bending moment exceeds the compression due to weight.

Example 2.13 Inertia loading in a mechanism



When the net acceleration of particles on the beam is not perpendicular to the beam, the axial component causes axial tension / compression, and the normal component causes shear forces and bending moments.

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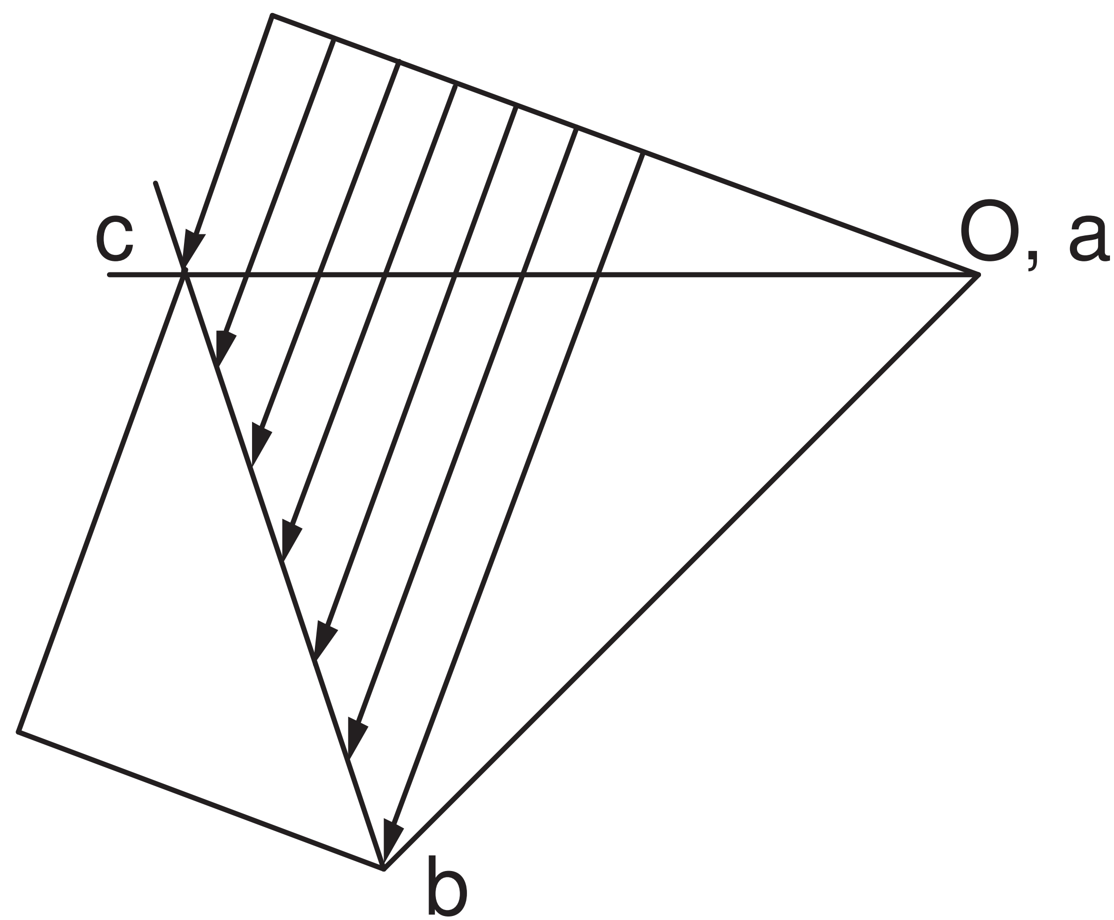
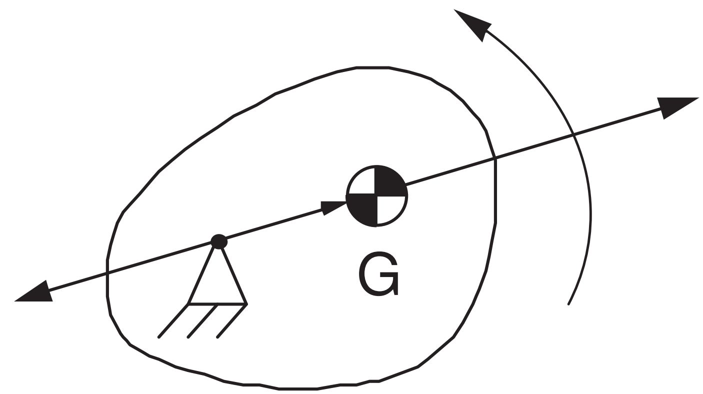


Acceleration Diagram

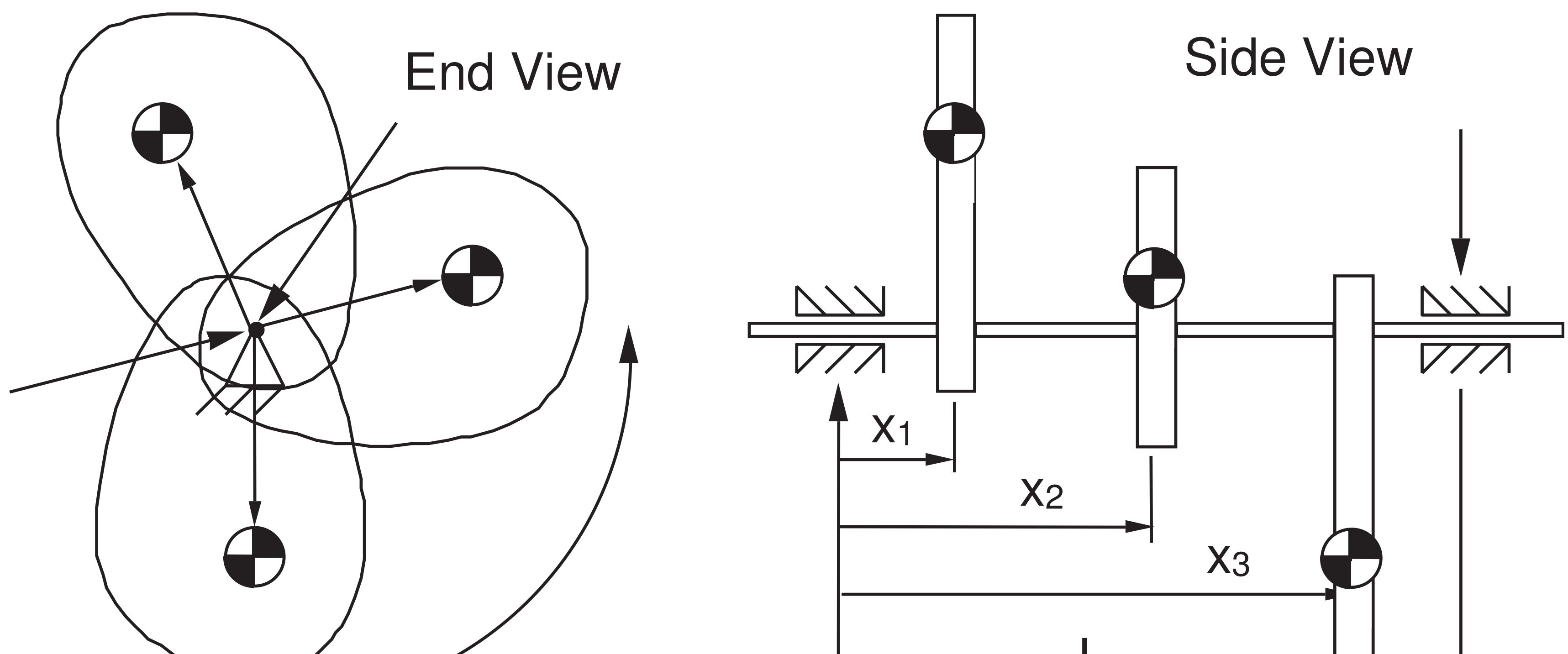
Now try Questions 1-4 on Examples Paper 3

2.9 BALANCING SIMPLE ROTORS

1 Body rotating in a plane



2 Several bodies rotating on a rigid shaft



Static Balance

Using d'Alembert:

The shaft is said to be ‘statically’ balanced if

(2.29)

i.e. if the overall centre of gravity G lies on the axis as per (2.28). If the shaft is mounted horizontally in frictionless bearings, it will rest in equilibrium in any angular position. This can theoretically be achieved ‘statically’

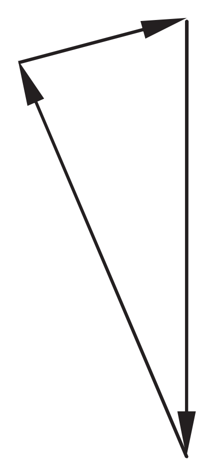
When a simply supported shaft is statically balanced,

(2.30)

i.e there is no <u>net</u> bearing force – but it might need a couple (equal and opposite forces) from the two bearings.

<u>Methods for achieving static balance</u>

1 If there are three or more bodies which can be rotated on the shaft, arrange them such that the polygon of $M_i\mathbf{R_i}$ closes:



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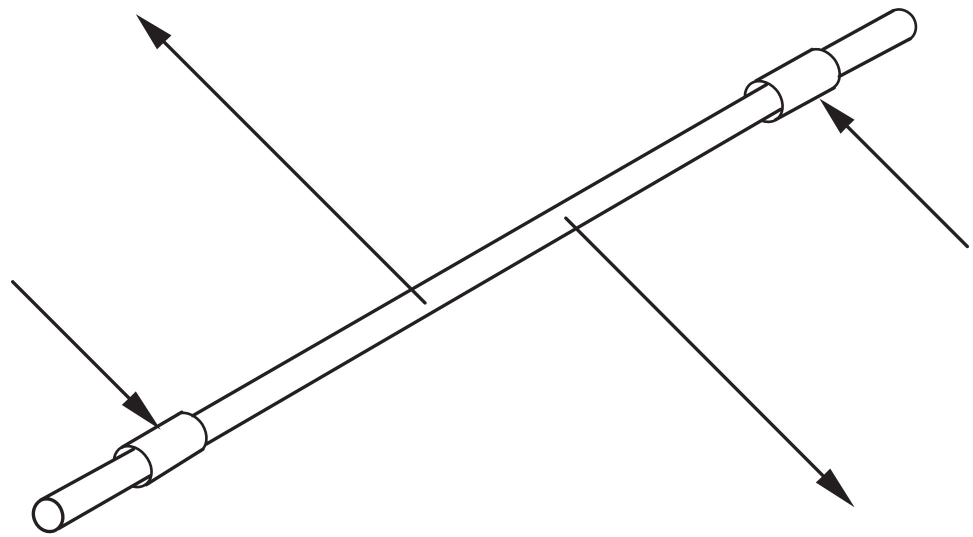
2. If the bodies cannot be moved, add a counterbalance mass of m_b at radius R_b such that:

3. For final balancing, remove a mass equivalent to $M\mathbf{R}_{\mathbf{G}}$ by machining or drilling.

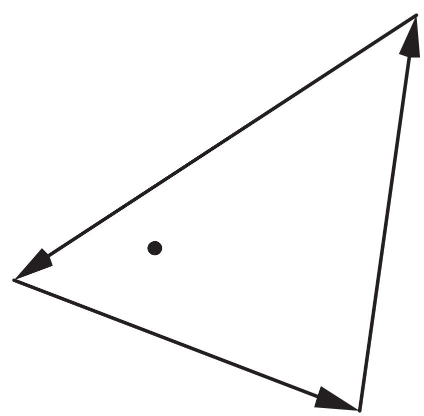
Dynamic Balance

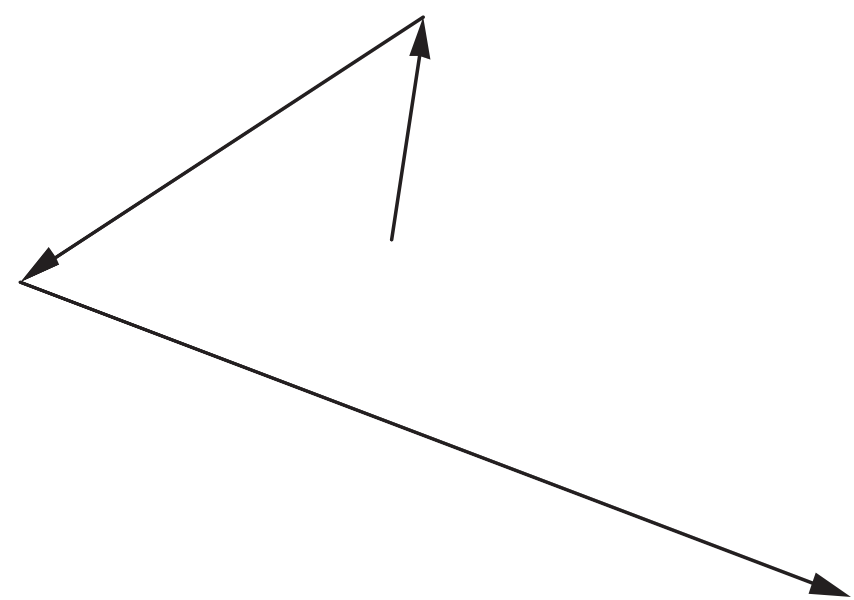
For a statically balanced shaft, the bearing forces sum to zero but are not individually zero ($\mathbf{F_A} = -\mathbf{F_B}$). In a simply supported shaft they are equal and opposite forces opposing the d'Alembert couple caused by the shaft's mass (which rotates with the shaft).



To achieve dynamic balance, i.e. $\mathbf{F_A} = \mathbf{F_B} = \mathbf{0}$, the net moment of the d'Alembert forces about any point must be zero.

To achieve dynamic balance, add 2 counterweights to close the moment polygon while still maintaining static balance:





Notes

(i) For static balance

(ii) For dynamic balance

(iii) In practice, friction prevents 'static balance' from being carried out statically. Therefore spin the shaft, measure the vibration of the casing, check the phase with the shaft position, the add/remove mass in 2 places.

(iv) Limited dynamic balancing can be achieved for a shaft with three rotors if they can be rotated and arranged in any order. Static balancing is first done using method (1) above. If the rotors are equally spaced, the out-of-balance force can be found by considering the mass of the centre rotor to be equally distributed between the outer two:

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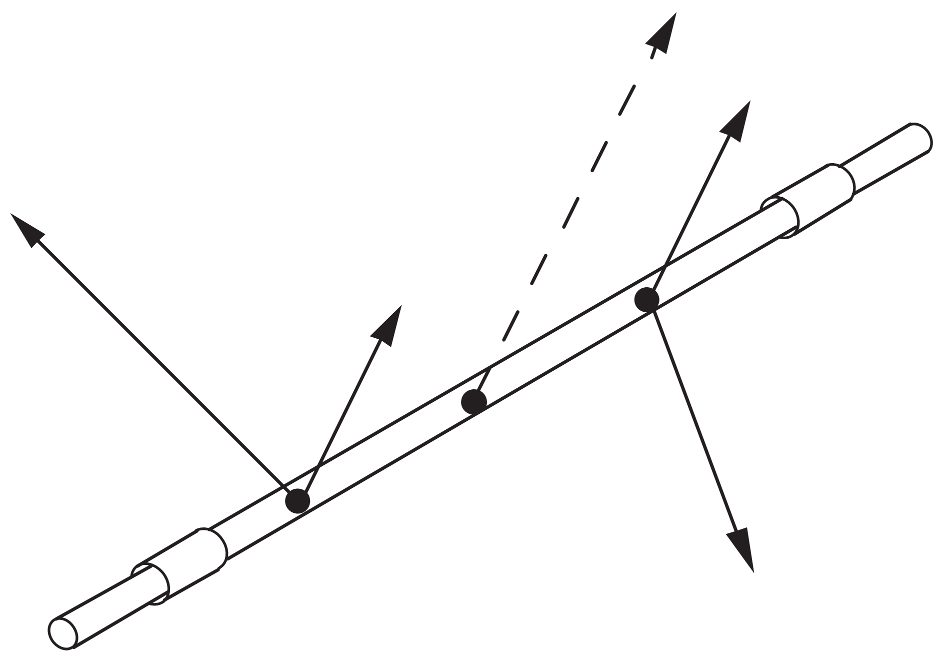

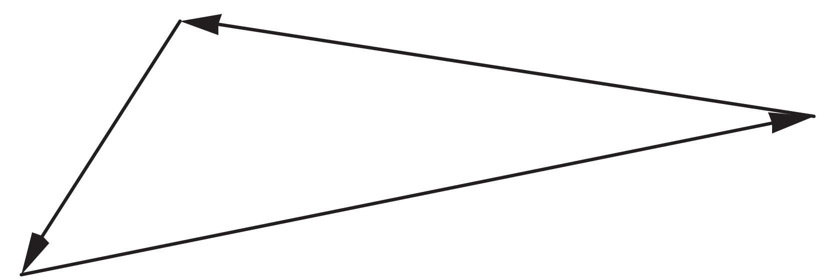

(v) Real balancing of shafts is much harder than this, because shafts (and bearings) are flexible. Perfect dynamic balance is never possible in practice, even with a large number of balancing stations.

Now try Questions 5-6 on Examples Paper 3