A model for small oscillations of a pressurised, elastic, spherical shell

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Abstract
This paper presents a novel model for the small oscillations of a pressurized, elastic, spherical shell. The shell has three features: a pressure difference across the skin; a thin, tensioned membrane; and a double curved interfacial surface. An analytical solution for the natural frequencies and mode shapes, incorporating the inertia both of the shell and of the surrounding fluid, is derived. When the membrane tension is set to zero, the results converge to the analytical solution for a spherical shell, and when the skin elasticity is neglected, the results converge to the constant-tension solution of a bubble.

Keywords: pressurised shell, elastic membrane, vibration modes

1 Introduction
This paper describes a new model that combines consideration of an elastic spherical shell, the influence of skin pre-tension, and the inertial effect of internal and external fluids. Example of such shells are sports balls, biological organs, airbags and balloons. There is a long tradition of analytical models of spherical shell vibrations, and the approach taken in this paper builds on some established models.

2 Existing Models
A well-known model of elastic spherical shell vibrations is based on Love’s deep-shell equations [1] and Lamb’s study on a thin spherical shell [2]. This model is unsuitable for application to the vibrations of pressurized, elastic, spherical shells as it does not account for: (a) the pre-tension in the skin that results from the pressure difference across the surface, that is, the initial stressed state; and (b) the influence of the inertia of the internal and external fluids on the radial momentum of the balloon. These two effects are present in Grinfeld’s model of the small oscillations of a spherical soap bubble [3], which extends Lamb’s solution [4] for the oscillation of a fluid drop to include the inertia of the soap film. This model is, however, unsuitable for the vibration of an elastic shell, as it assumes constant uniform and isotropic skin tension (surface tension) in line with Laplace’s classical model of capillarity. This means that the changes in stress state in the shell due to elastic deformation are neglected. For this reason, a new model is developed that draws on elements of these two existing models.

3 Model Formulation
This formulation considers linearized, axisymmetric, normal modes that are harmonic in time with angular frequency $\omega$. There is no loss of generality in focussing on axisymmetric modes for convenience, since a non-axisymmetric normal mode can be written as a linear combination of rotations of an axisymmetric mode of the same frequency.

For simplicity, we assume that the fluids inside and outside the shell are incompressible and inviscid. The effect of compressibility is negligible provided that $\omega \ll c/R$, where $c$ is the acoustic wavespeed and $R$ is the shell radius. The effect of fluid viscosity is negligible provided that the Reynolds number is much greater than one. Euler’s equations for incompressible, inviscid flow then apply, and axisymmetric solutions in spherical coordinates are assumed. The kinematic condition at the fluid-shell interfaces requires that the shell’s normal velocity component is equal to both the internal fluid’s normal velocity component and the external fluid’s normal velocity component.

The motion of the thin, spherical shell results from the combination of three effects: elastic thin-shell deformations; the pre-tension in the shell that results from the initial pressure difference across the surface; and the influence of the inertia of the internal and external fluids on the momentum of the shell. The elastic shell deformations are captured in [1,2], and we
now consider how to incorporate the tensile and inertial effects into these equations.

The shell is considered to be under an initial state of isotropic tension \( T \). When the shell deforms from its equilibrium position, the effect of this tension is a radial restoring stress proportional to the tension and the change in curvature due to deformation. As the tension is isotropic, no force occurs in the meridional direction. As the fluids are assumed inviscid, there are no fluid inertia effects in the meridional direction; this influence is only in the radial direction. The radial restoring stress due to the fluid inertia is simply the difference in pressure between the internal and external fluids.

The small displacements in the radial and meridional directions, \( u \) and \( v \) respectively, are assumed to be of the form:

\[
\begin{align*}
\dot{u} &= U P_n(\cos \phi) e^{i t} \\
\dot{v} &= V Q_n(\cos \phi) e^{i t}
\end{align*}
\]

where \( P_n(\cos \phi) \) and \( Q_n(\cos \phi) \) are the Legendre functions of the first and second kind, respectively. Then the governing equations of a shell of thickness \( h \), density \( \rho \), Young’s modulus \( E \) and Poisson’s ratio \( \nu \) are:

\[
\begin{align*}
\frac{Eh}{(1-\nu^2)R^2} [ -n(n+1)V + 2U ] &+ \frac{(n^2 + n - 2)TU}{R^2} = A \rho h \omega^2 U \quad (1a) \\
\frac{Eh}{(1-\nu^2)R^2} [ -n(n+1)V + (1-\nu)V + (1+\nu)U ] &+ \rho h \omega^2 V = 0 \quad (1b)
\end{align*}
\]

where \( n = 0, 1, 2, ... \) is the mode number. The non-dimensional parameter \( A \) represents the effective shell inertia:

\[
A = 1 + \frac{\rho_e R}{\rho_h (n+1)} + \frac{\rho_i R}{\rho_h n} \quad (2)
\]

where \( \rho_e \) and \( \rho_i \) are the densities of the external and internal fluids, respectively.

The natural frequencies of the shell are then the (positive) solutions of \( \omega \). For each value of \( n \), two natural frequencies exist. These natural frequencies correspond to different mode shapes with varying ratios \( U/V \) of radial and meridional motion of the skin. The lower branch has more radial motion and the upper branch less, but both branches involve elastic coupling between the two directions of motion.

Setting the skin tension to zero \( (T = 0) \) and removing the effects of the internal and external fluids \( (A = 1) \) recovers the solution for an elastic shell given in [1,2]. The solution for a pre-tensioned elastic shell with no surrounding fluids is found by setting \( A = 1 \). Removing the elasticity in the skin \( (E = 0) \) results in a single governing equation, which can be rearranged to obtain Grinfeld’s solution for the natural frequencies of a soap bubble. The removal of the elasticity term results in only one natural frequency per modenumber \( n \), with the corresponding modeshapes consisting of purely radial skin motion.

4 Conclusions

This paper has described the formulation of a model for the vibrations of an elastic, spherical shell, subject to internal pressure. The model formulation draws on elements of two existing models: the well-known elastic shell model based on Love’s deep shell equations and Lamb’s study on a thin spherical shell; and a model for soap-bubble vibrations that incorporates fluid effects.

References


