Minimizing the Self-weight Deflection of Tensegrity Structures

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Summary

In this paper, we consider optimizing the design of *kinematically indeterminate* tensegrity structures. These tensegrity structures will have modes of deformation where the stiffness will scale linearly with a typical level of tension carried by the structure. These modes tend to be the softest modes of deformation, and can lead to significant deformations due to self-weight: this tends to be most noticeable for highly symmetric structures, where, for instance, significant oblation may occur if the structure is resting on a flat surface. We will explore how these deformations can be optimized through material choice. We will also consider optimal choices where, for aesthetic reasons, the compression members are slender.

Keywords: kinematically indeterminate tensegrity structures, optimization of tensegrity structures, self-weight deflections.

1. Introduction

We were inspired to write this paper after playing a very minor advisory role in the design of the tensegrity structure shown in Fig.1. A concern at the design stage was that the self-weight deflection of the structure would detract from the aesthetic appeal of the highly symmetric structure. After seeing the completed structure, we decided to investigate more carefully the factors that would minimize the self-weight deflection of this type of structure.

Fig. 1: A sculptural tensegrity designed and built by D. Burnett and J. Wythe. The struts are of Aluminium, and have a length of 2500mm, and the cables are of steel.

The 'ideal' version of the tensegrity shown in Fig.1 is shown in Fig.2. This structure has the rotational symmetries of a regular icosahedron, which is isomorphic to the alternating group on 5 symbols, A_5. It may be found in Connelly's online catalogue of simple symmetric tensegrity structures (Connelly and Back, 1998) at http://mathlab.cit.cornell.edu/visualization/tenseg/ as Example 6.1 of Conjugacy Class 6 of Group A_5. The structure consists of three *orbits* of structural members, where the members of each orbit are equivalent to each other under a symmetry operation. In total there are 90 cables and 30 struts connecting 60 nodes.
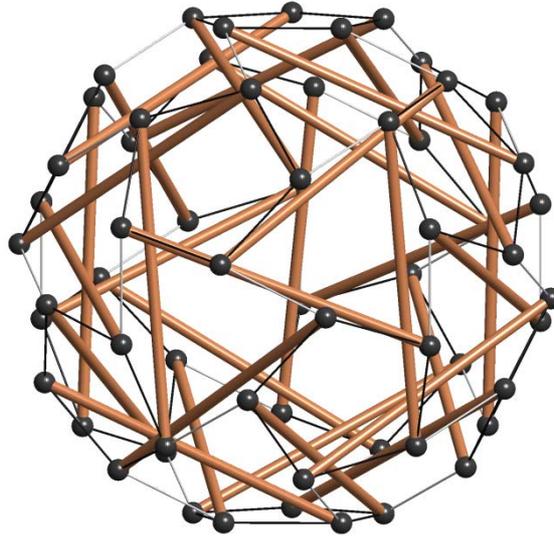
Fig. 2: The 'ideal' form of the tensegrity. The structure has the rotational symmetries of a regular icosahedron. It consists of one orbit of struts (coloured brown), and two orbits of cables (coloured grey and black). One orbit of cables form pentagons, while the other connects these pentagons together; the cables have the same graph as the edges of a truncated dodecahedron.

Using the symmetry methods described by Ramar and Guest (2006) and Ramar (2009), it is possible to show that when the structure is prestressed, it must have the same tension coefficient (tension/length) in members of both orbits of cables, and the ratio of the tension coefficient in the cables to that in the struts is $2.6182 : -1$. If we choose to have all cables the same length and struts of length 2500 mm, then all cables must have length 892.1 mm. The tensegrity then has a single state of self-stress, s = 1. Using Calladine's extended Maxwell's rule (Calladine, 1978) for a three-dimensional tensegrity framework composed of $b = 120$ bars and $j = 60$ joints,

$$b - 3j + 6 = s - m,$$

(1)

we find that there are $m = 55$ infinitesimal mechanisms: modes of deformation that, to first order, do not require any member to change length. The methods described by Ramar (2009) to generate this tensegrity guarantees that each of these modes is given positive stiffness when the structure is prestressed. However, Guest (2011) showed that for these kinematically indeterminate tensegrity structures, made from materials with a small yield strain (e.g. metals with a yield strain of less than 1%) the modes that correspond to infinitesimal mechanisms will dominate the response of the structure to loading, as they have a far smaller stiffness than any other modes of deformation. Guest (2011) also showed that these modes have stiffness that is proportional to the tension coefficient carried by the members, and we will make use of this for our simple model in Section 2.

The layout of the paper is as follows. Section 2 describes a simple model to capture the important parameters affecting the self-weight deflection of tensegrity structures. Section 3 provides a computational model to find the self-weight deflection of these structures, and shows how the choice of materials and sections influences the optimal design for the self-weight deflection. Section 4 concludes the present work.

2. A Simple Model

In this section we will generate a simple model that will attempt to capture the important parameters in calculating the self-weight deflection of a prestressed kinematically indeterminate tensegrity structure. We assume that the basic design of the tensegrity is fixed, and hence the ratio of the tension in the cables and compression in the struts is fixed. We make an initial assumption that the weight, W, of the structure is dominated by the weight of the struts; that is, we are neglecting the weight of cables and any fittings at the nodes. In fact we will see that this is a good assumption as long as the struts are slender. Consider that the struts have length l, cross sectional area A, and are made of a material with density ρ. Then, with acceleration due to gravity g, we can write

$$W \propto g\rho lA \tag{2}$$

Further, we assume that behaviour of the structure is dominated by the modes that are infinitesimal mechanisms, which Guest (2011) showed to have stiffness S proportional to the tension coefficient in the members. If the level of prestress is such that the cables carry a tension t, then we can write

$$S \propto \frac{t}{l} \tag{3}$$

Thus, we can calculate the deflection $d = W/S$,

$$d \propto \frac{\rho g l^2 A}{t} \tag{4}$$

The simple conclusion from this is that, for a given model, it is always possible to decrease the self-weight deflection by simply increasing the level of prestress. However, this will ultimately be limited, either by yielding of the cables or struts, or buckling of the struts. Here, we will assume that the level of tension is limited by Euler buckling of the struts. If the struts have a radius of gyration r_g (and hence a second moment of area Ar_g^2), and are made of material with a Young Modulus E, the Euler buckling load is $\pi^2 EI/l^2$. The cable tension t must be proportional to this, and so

$$t \propto \frac{EAr_g^2}{l^2} \tag{5}$$

Substituting this back into Eq.4 gives

$$d \propto \frac{\rho g l^4}{Er_g^2} \tag{6}$$

Which we can write in non-dimensional form as

$$\left(\frac{d}{l}\right) \propto \left(\frac{\rho g l}{E}\right)\left(\frac{l}{r_g}\right)^2 \tag{7}$$

Thus, in order to reduce the self-weight deflection for a structure with a given size l, we have to: (1) choose a material that maximises the ratio (E/ρ); and (2) decrease the slenderness (l/r_g).

2.1   Material optimisation

Fig.3 shows an Ashby bubble chart comparing Young's Modulus E and Density ρ for different materials. In order to maximise the ratio (E/ρ) we are looking for materials towards the top-left corner of the plot. Thus we see that steel and aluminium are likely to have a similar performance in this application (as would bamboo), whereas carbon fibre should give reduced self-weight deflection. Interestingly, the plot suggests that technical ceramics might prove to be an optimal material choice for the struts, but we have not explored that here.
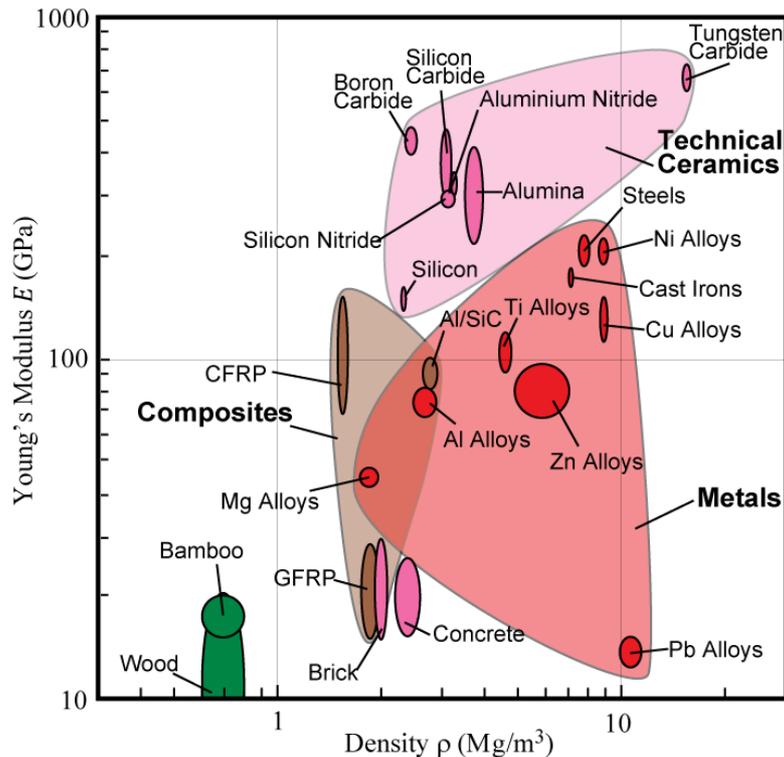
Fig. 3: An Ashby bubble chart comparing Young's Modulus E and Density ρ for different materials. Data from Cambridge Engineering Selector software, 2011, courtesy of Granta Design Ltd, Cambridge UK

2.2   Strut optimisation

It is interesting to note that Eq.7 doesn't depend on the area of the strut, simply on its radius of gyration. Assuming that the struts have a circular cross-section with diameter D_0, then the radius of gyration will lie in a range between $r_g = D_0/4$ for a solid cross-section and $r_g = D_0/2\sqrt{2}$ for a thin-walled cross-section, with the thin-walled section giving (unsurprisingly) a smaller self-weight deflection for a given D_0. Although Eq.7 would suggest that increasing the diameter D_0 of the struts will always decrease the self-weight deflection, there are limits to this. The physical limit is that, at some point, the struts will yield rather than buckle; but in practice their will be an aesthetic limit well before this point, where the struts are no longer slender, as, for example, we will see later in Fig.4 (b). We will return to the aesthetic trade-off in the design of these structures in Section 3.

3.   A Computational Model

To compare the self-weight deflection of tensegrity structures in more detail, we made use of simple linear computational model to calculate deflections for different materials and different levels of stress. We used the *tangent stiffness matrix* formulation to understand the mechanics of the tensegrity structures. Guest (2006) developed a novel derivation of the tangent stiffness matrix

that makes clear the role that a *stress matrix* plays in the stiffness of a self-stressed framework. The total tangent stiffness $\mathbf{K}$ can be written as

$$\mathbf{K} = \mathbf{A}\widehat{\mathbf{G}}\mathbf{A}^{\mathrm{T}} + \mathbf{S} \tag{8}$$

where $\mathbf{A}$ is an *equilibrium matrix*, which depends on direction cosines of the members, $\widehat{\mathbf{G}}$ is a diagonal matrix of *modified axial stiffness*, and $\mathbf{S}$ is a *large stress matrix*, which depends on the tension coefficients. We assume that five nodes forming one pentagon are attached to the ground, and the weight of the members is then applied as a loading at the other nodes.

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Table 1: Material Properties

	High Yield Strength Steel	Aluminium Alloy	Carbon Fibre
Yield Stress, Y (N/mm^2)	1470	400	900
Young's Modulus, E (N/mm^2)	2.0 x 10^5	0.7 x 10^5	0.95 x 10^5
Specific Weight, ρg (N/mm^3)	78.50 x 10^{-6}	27.0 x 10^{-6}	16.0 x 10^{-6}

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The cables are designed using high yield strength steel, while for the struts, three different materials, shown in Table 1, are considered. The structure is designed as follows. First a cable area A_{cable} is assumed, and the limiting tension found for avoid yield,

$$t = Y_{\mathrm{steel}}A_{\mathrm{cable}} \tag{9}$$

The cable tension coefficient is thus given by $\hat{t}_{\mathrm{cable}} = t/l_{\mathrm{cable}}$, and, for equilibrium, as described in the Introduction, the strut tension coefficient must be $\hat{t}_{\mathrm{strut}} = \hat{t}_{\mathrm{cable}}/-2.6182$ and the strut tension $t_{\mathrm{strut}} = \hat{t}_{\mathrm{strut}} \times l$. Note that the strut tension is negative, as it is a compressive. The strut is designed to avoid buckling for the compressive force.

$$\frac{\pi^2 E A r_g^2}{l^2} = -t_{\mathrm{strut}} \tag{10}$$

In Eq.10 the unknown term Ar_g^2, the second moment of area of the cross section, can be written using the cross sectional dimensions of the strut. For a solid circular section of outer diameter D_o, $Ar_g^2 = \pi D_0^4/64$. In case of a tubular strut, we choose a limiting wall thickness of $D_o/60$ to avoid local buckling (Gresnigt, 2010), and $Ar_g^2 = \pi(D_o^4 - D_i^4)/64$, with D_i the inner diameter. Note that, for simplicity, we do not consider yielding of the strut, nor reduced buckling strength due to imperfections – but these matters are will only have a significant effect for stocky struts that are unlikely to be used for a tensegrity structure. Once the size of members and level of tension is known, the tangent stiffness and the weight can be found, and hence the deflection calculated.

As an example, consider the two designs of tensegrity shown in Fig.4, both of which have $l = 2500$ mm and solid steel struts. The structure in Fig.4(a) has cables 4.0 mm in diameter, struts of diameter 33.61 mm, and a minimum self-weight deflection with maximum prestress of 83.04 mm. An alternate design in Fig.4(b) has cables 20.0 mm in diameter, struts of diameter 75.14 mm giving a self-weight deflection 17.62 mm. There is a fundamental trade-off between minimizing the diameter of the struts, and minimizing the self-weight deflection, a point to which we will return in the Conclusion.
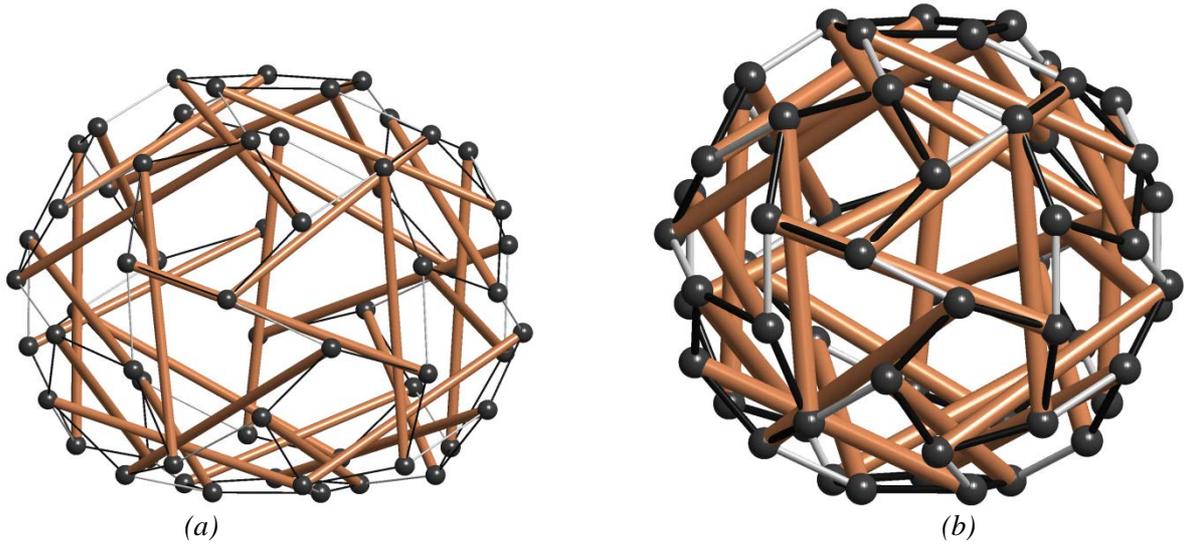
Fig. 4: Self-weight deflections of the example tensegrity structure with solid steel rods supported on a horizontal plane: (a) large deflection for slender struts; (b) reduced self-weight deflection with higher prestress and stockier struts.

Figs.5 and 6 show the results for designs with varying cable diameters for different materials and cross-sections. This confirms that thin-walled carbon fibre struts give the lowest self-weight deflections amongst the materials and cross-sections considered. Fig.5 shows the data in simple form, whereas Fig.6 shows the data in the non-dimensional form suggested by the simple model in Section 2. Eq.7 suggests that the data should fall onto a single line of slope 2 on the log-log plot, and this indeed turns out to be the case for slender struts, with large l/r_g. However, for stocky struts, particularly those made from carbon fibre, the (neglected) weight of the cables becomes significant, and the results deviate from the simple model.
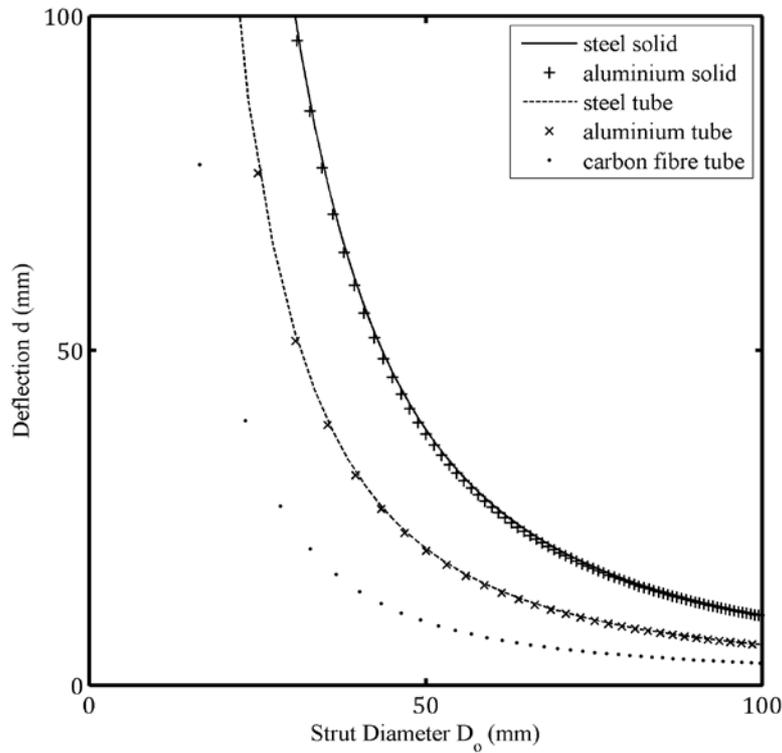
Fig. 5: Minimum self-weight deflection for varying strut diameters, for different materials and cross-sections.
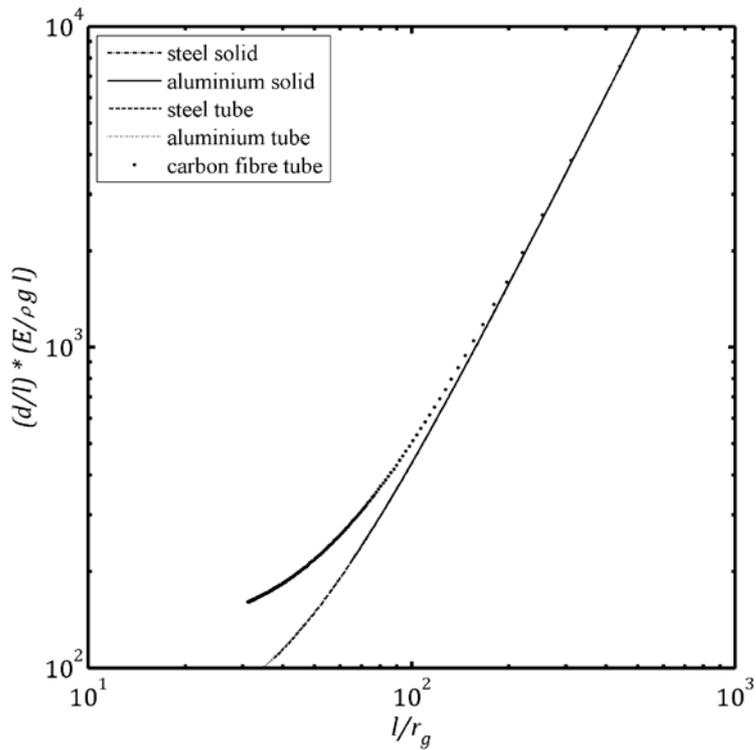
Fig. 6: Minimum self-weight deflection plotted in a non-dimensional form suggested by Eq.7, for varying strut diameters, for different materials and cross-sections.

As a final example, we have modelled the self-weight deflection of the sculpture described in the Introduction. The structure is constructed with 4.0 mm diameter wound steel cables, and aluminium alloy tubular struts with outer diameter 63.0 mm, and wall thickness 1.6 mm. A tension of 1000.0 N is carried by the cables. Using the computational model discussed above, the self-weight deflection is found to be 197.52 mm, which is 5.23% of the height of structure. We have not attempted to measure this deflection, but the deflected configuration found in the study is compared with the real tensegrity structure in Fig.7.
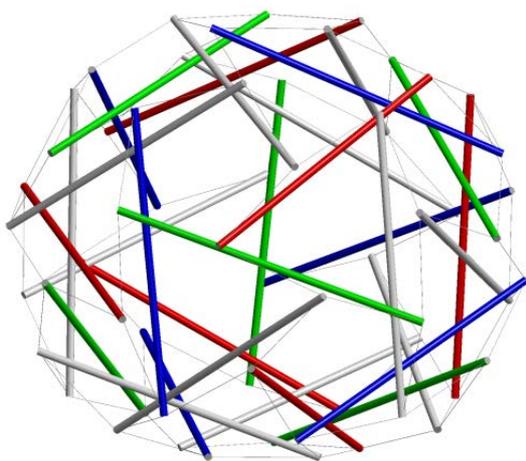
(a)   *(b)*

Fig. 7: Self-weight deflection: (a) computational model (b) actual structure.

4. Conclusion

We have shown that increasing the prestress in a kinematically indeterminate tensegrity structure will decrease the deflection under load, and that the key to reducing self-weight deflections is to design the structure so that the prestress can be increased without increasing the weight unduly. In particular, choosing a material for the struts that maximises (E/ρ), and using thin-walled members, are straightforward steps. Beyond this there is a fundamental aesthetic trade-off between reducing the self-weight deflection, and reducing the slenderness of the struts.

This paper has only considered the design of the struts, but there is also scope for optimising the design of the cables. In particular, a lightweight material such as Kevlar might reduce the required mass of the cables, if suitably light end-fittings can be designed.

5. Acknowledgement

6. References

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