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A symmetry-extended mobility rule

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Abstract

A symmetry-extended mobility rule for mechanical linkages is presented, in which the conventional mobility criterion for a linkage is subsumed and strengthened by an equation that predicts symmetries, as well as numbers, of mobilities and states of self stress.

Keywords: Symmetry; Mobility

1 Introduction

The mobility criterion, a simple generic counting relationship to calculate the degrees of freedom of a mechanical linkage, is a familiar concept in mechanism theory, attributed by Hunt [1] to Grübler [2,3] or Kutzbach [4]. One form, on which we shall build in this paper, is given by e.g. Hunt [1]: the relative degrees of freedom, or the *mobility*, m, of a mechanical linkage consisting of n bodies connected by g joints, where joint i permits f_i relative freedoms, is

$$m = 6(n-1) - 6g + \sum_{i=1}^{g} f_i \tag{1}$$

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The meaning of this equation is clear: in the absence of connections between bodies, and relative to one body considered as a reference, the other bodies have 6 freedoms each; freedoms are then removed by the $\sum_{i=1}^{g}(6 - f_i)$ constraints at the joints.

It is the purpose of the present note to point out that a more specific form of mobility rule can be found by considering (1) in the light of not only the *numbers* of bodies, joints and freedoms, but also their *symmetries*. Given the *reducible representations* of bodies, joints and freedoms, which we shall show are easily calculated by counting the structural components left unshifted by various symmetry operations, the algebraic formula (1) appears as an aspect of a more general relation which can give useful symmetry information about possible mobility. This extension of an algebraic to a group-theoretical relation parallels developments in structural engineering and chemistry, where Maxwell's rule for the rigidity of a structure, and Euler's polyhedral theorem relating numbers of vertices, edges and faces, have both been shown to have powerful symmetry counterparts [5,6]. Previous authors have used simple symmetry relations to find or classify overconstrained linkages (e.g. [7–9]), but have not placed their work in a general group theoretical context. Here we give a formulation that applies to all linkages of all symmetries.

The paper will proceed as follows. First, a generalised form of (1) will be developed, in order to consider possible states of self-stress in the mechanical linkage, as well as mobility, in a similar fashion to Calladine's generalisation of Maxwell's rule [10]. Then, a symmetry-extended form of this generalised mobility rule will be developed. Application of this extended relation requires an understanding of how joint freedoms transform under symmetry operations, and this will be developed for the one-degree-of-freedom lower pairs. A simple example will be given that shows the advantage of a symmetry treatment over simple counting.

2 A generalised mobility rule

There are many cases for which the generic mobility criterion given in (1) is not actually applicable, e.g. for particular geometric configurations where constraints at the joints are not independent. A form of (1) that *does* apply in all cases is

$$m - s = 6(n-1) - 6g + \sum_{i=1}^{g} f_i \tag{2}$$

where s is the number of independent states of self-stress that the mechanical linkage can sustain: a state of self-stress is a set of internal forces in the linkage in equilibrium with zero external load. s can be considered equivalently as the number of *overconstraints* — independent geometric incompatibilities, or misfits — that are possible for the linkage. It is this generalised mobility criterion, (2), for which we shall develop the symmetry-extended form.

Equation (2) can most easily be demonstrated by considering a *compatibility matrix* $\mathbf{C}$ and an *equilibrium matrix* $\mathbf{A}$ for a mechanical linkage [11]. The compatibility matrix gives an instantaneous relationship $\mathbf{Cd} = \mathbf{e}$ between relative infinitesimal motions of the rigid bodies (rotations and displacements), written as a vector $\mathbf{d}$, and independent 'strains' at joints, written as a vector $\mathbf{e}$. $\mathbf{d}$ will have $6(n-1)$ components (considering one body as a reference), while $\mathbf{e}$ will have $\sum_{i=1}^{g}(6-f_i)$ components, one for each constraint at each joint. The nullspace of $\mathbf{C}$ corresponds to solutions of $\mathbf{Cd} = \mathbf{0}$, and has dimension

$$m = 6(n-1) - r, \tag{3}$$

where r is the rank of $\mathbf{C}$ [12]. If there are at least as many freedoms as constraints, $6(n-1) \geq \sum_{i=1}^{g}(6-f_i)$, and the matrix is of full rank, then $r = \sum_{i=1}^{g}(6-f_i)$, and (1) is correct — in fact, careful statements of (1) state that it applies when constraints are independent, which implies these conditions.

An argument based on virtual work shows that an equilibrium matrix $\mathbf{A} = \mathbf{C}^{\mathrm{T}}$ describes the relationship $\mathbf{Ar} = \mathbf{p}$ between the internal forces at the joints, written as a vector $\mathbf{r}$, and forces and moments applied to the bodies, written as a vector $\mathbf{p}$, where $\mathbf{r}$ and $\mathbf{p}$ are work-conjugate to $\mathbf{e}$ and $\mathbf{d}$, respectively. The nullspace of $\mathbf{A}$ (the left-nullspace of $\mathbf{C}$) corresponds to solutions of $\mathbf{Ar} = \mathbf{0}$, i.e. states of self-stress, and has dimension

$$s = \sum_{i=1}^{g}(6-f_i) - r, \tag{4}$$

If there are at least as many rigid-body freedoms as joint constraints, $6(n-1) \geq \sum_{i=1}^{g}(6-f_i)$, and the matrix is of full rank, then $r = \sum_{i=1}^{g}(6-f_i)$, and $s = 0$, which is implied in (1).

Eliminating r between (3) and (4) gives the generalised mobility rule, (2). Typically, r can only be determined by full knowledge of the instantaneous geometry of the mechanical linkage, although often further insight can be obtained if the geometry can be recognised as having some special property, for instance belonging to a specific category of screw system [1, Chapter 13].

The next section will develop a symmetry-extended mobility criterion that compares not only the total number, but also the symmetries, of the mobility and the states of self-stress.

3 A symmetry-extended mobility rule

3.1 Representations

This section will develop a symmetry-extended mobility rule using the language of *representations*. The representation of an object will be written as Γ(object), and describes its symmetry in a relevant point group. A point group consists of a set of symmetry operations S, and the objects we are considering are (sets of) points, vectors representing translations and forces, pseudo-vectors representing rotations and moments etc. The representation Γ(object) collects the *character* $\chi(S)$ of such sets under S, i.e. the trace of the matrix that relates the set before and after the application of S.

The development of (2) can be repeated in a symmetry-extended form by examining the difference between the symmetries of the freedoms of the rigid bodies that make up the mechanical linkage, and the symmetries of the constraints. The development will broadly follow that outlined for the symmetry-adapted Maxwell rule [5], and a reader unfamiliar with the language of representations will find it presented there in a mechanics context.

In the language of representations, the extended mobility criterion can be written as

$$\Gamma(m) - \Gamma(s) = \Gamma(\text{relative body freedoms}) - \Gamma(\text{hinge constraints}) \tag{5}$$

where $\Gamma(m)$ and $\Gamma(s)$ are the representations of the mobility, and the states of self-stress, respectively. The other terms are self-explanatory; they will be expanded using the notion of a *contact polyhedron* C, with 'vertices' at bodies, and 'edges' through joints. (Strictly speaking, C is not always a polyhedron, as for many mechanical linkages it will not be possible to define the *faces* of C, but we will nonetheless continue to use this terminology.) All representations will be calculated in $G(C)$, the point group of C, rather than the point group of the linkage from which it was derived. Symmetries of C must preserve the axes of the hinges, but within this restriction C is constructed to be maximally symmetric. C may have higher symmetry than the actual object — the actual shapes of bodies, or the actual physical positions of hinges along hinge lines, have no effect on the symmetry of the mobility.

3.2 Relative body freedoms

The relative body freedoms are the freedoms of all the bodies of the mechanical linkage, in the absence of connections, and taken relative to one body considered as a reference. Their representation can be written as that of the total freedoms of the bodies, minus that of the rigid body motions:

$$\Gamma(\text{relative body freedoms}) = \Gamma(\text{body freedoms}) - \Gamma(\text{rigid body motion}), \quad (6)$$

where

$$\Gamma(\text{rigid body motion}) = \Gamma_T + \Gamma_R, \quad (7)$$

and Γ_T and Γ_R are the representations of rigid-body translations and rotations, and can be read off from point-group theory tables, e.g. [13]. In terms of the contact polyhedron C, the three rotational and three translational freedoms per body span

$$\Gamma(\text{body freedoms}) = \Gamma(v, C) \times (\Gamma_T + \Gamma_R) \quad (8)$$

where $\Gamma(v, C)$ is the *permutation representation* of the vertices of C. A permutation representation of a set has character $\chi(S)$ equal to the number of elements of the set left in place by the operation S.

3.3 Hinge constraints

An expansion of the representation of the hinge constraints is less straightforward. As in the counting rule (1), we expand Γ(hinge constraints) as the representation of constraints imposed by hypothetical rigid joints, minus that of the actual freedoms at the joints:

$$\Gamma(\text{hinge constraints}) = \Gamma(\text{rigid joints}) - \Gamma_f, \quad (9)$$

where Γ_f is the representation of joint freedoms. We will show later that the constraint representations can be written as

$$\Gamma(\text{rigid joints}) = \Gamma_\parallel(e, C) \times (\Gamma_T + \Gamma_R) \quad (10)$$

where $\Gamma_\parallel(e, C)$ is the representation of a set of vectors along the edges of C. The proof of (10) follows immediately from the description of the six freedoms between two rigid bodies, which is presented next; the proof itself will be given in Section (3.6).

3.4 General form of the symmetry-extended mobility rule

Substituting (6)–(10) into (5) gives the symmetry-extended mobility rule

$$\Gamma(m) - \Gamma(s) = \Gamma(v, C) \times (\Gamma_T + \Gamma_R) - \Gamma_\parallel(e, C) \times (\Gamma_T + \Gamma_R) - (\Gamma_T + \Gamma_R) + \Gamma_f \quad (11)$$

or, more succinctly,

$$\Gamma(m) - \Gamma(s) = (\Gamma(v, C) - \Gamma_{\|}(e, C) - \Gamma_0) \times (\Gamma_T + \Gamma_R) + \Gamma_f \tag{12}$$

where Γ_0 is the totally symmetric representation, with $\chi(S) = 1$ for all S. Use of this rule requires the evaluation of Γ_f. As Γ_f may represent combinations of many different types of joint, there is little to be gained by attempting to write it in an explicit form. The calculation of Γ_f is straightforward in any particular case, and the following section shows how to do this.

3.5 Joint freedoms

The representation Γ_f can be found by calculating the character of the representation under each of the symmetry operations of the group $G(C)$. The present section will describe how these characters can be found.

A contribution to the character $\chi_f(S)$ occurs only when a joint is unshifted under the symmetry operation S. All joints are unshifted by the identity, and hence the character under the identity, $\chi_f(E)$, is simply the summation over all joints of the degrees of freedom at each joint. Under other symmetry operations, the contribution of each unshifted joint depends on the *type* of joint, and the relative *orientation* of the axis of the symmetry operation, the axes (if any) of the joint, and the directrix of the 'edge' in C. The calculation can be simplified, however, because the set of freedoms at any joint can be split into *revolute*, *prismatic*, and *screw* freedoms, and simple expressions for the characters of these individual freedoms are easily found, as detailed in the following sections. Of course, revolute and prismatic freedoms *can* be considered as special cases of screw freedoms, but in general a screw is not preserved under a reflection, which changes a right-hand to a left-hand screw, and so treating prismatic or revolute joints as screws may obscure the symmetry.

3.5.1 Revolute, prismatic and screw freedoms

The character of the freedom associated with a one-degree-of-freedom lower pair that is unshifted by a symmetry operation S depends upon the type of joint. For the three possible joint types we shall show that the character is given by:

$$\chi_{\mathrm{revolute}}(S) = \chi_{R_r}(S)\chi_{e_{\|}}(S), \tag{13}$$

where $\chi_{R_r}(S)$ is the character of R_r, a rotation about the axis of the revolute freedom, r, and $\chi_{e_{\|}}(S)$ is the character of a vector lying along the edge e of C;

$$\chi_{\mathrm{prismatic}}(S) = \chi_{T_p}(S)\chi_{e_{\|}}(S), \tag{14}$$

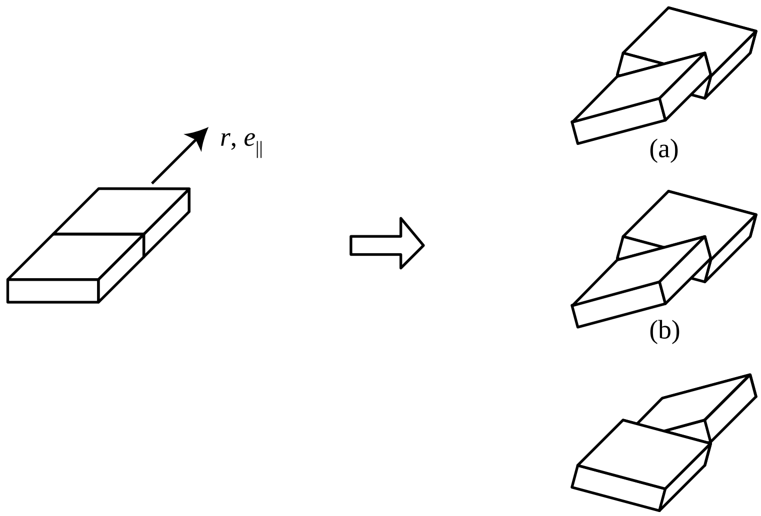


Figure 1: Examples of the character of the freedom associated with a revolute joint under symmetry operations. (a) A relative rotation between two bodies. (b) The same relative rotation followed by a rotation of the whole system by π about r: $\chi_{R_r} = 1$; $\chi_{e_\parallel} = 1$; and, by (13), $\chi_{\text{revolute}} = 1 \times 1 = 1$. (c) The same original relative rotation followed by a reflection in the plane perpendicular to r: $\chi_{R_r} = 1$; $\chi_{e_\parallel} = -1$; and, by (13), $\chi_{\text{revolute}} = -1 \times 1 = -1$.

where $\chi_{T_p}(S)$ is the character of a translation along the axis of the prismatic freedom, p;

$$\chi_{\text{screw}}(S) = \chi_{T_h}(S)\chi_{e_\parallel}(S) = \chi_{R_h}(S)\chi_{e_\parallel}(S), \tag{15}$$

where $\chi_{R_h}(S)$ is the character of R_h, a rotation about the axis of the screw freedom, h, and $\chi_{T_h}(S)$ is the character of a translation along h. As only proper operations are possible for a screw freedom, $\chi_{R_h}(S) = \chi_{T_h}(S)$.

The character under S of the total set of revolute, prismatic and screw freedoms of a linkage is then given by summing χ_{revolute}, $\chi_{\text{prismatic}}$ and χ_{screw} over all joints of the respective types that are unshifted by S.

Some examples demonstrating the behaviour of a hinge under symmetry operations are shown for a revolute hinge in Fig. 1, and for a prismatic hinge in Fig. 2.

Equations (13)–(15) can be proved straightforwardly by considering the limited set of symmetry operations S that may act on the joint without shifting it. Considering the joined bodies as structureless points, the assembly of two bodies, taken together with the hinge that connects them and with the corresponding edge of C, has maximum site symmetry $D_{\infty h}$ (when the edge

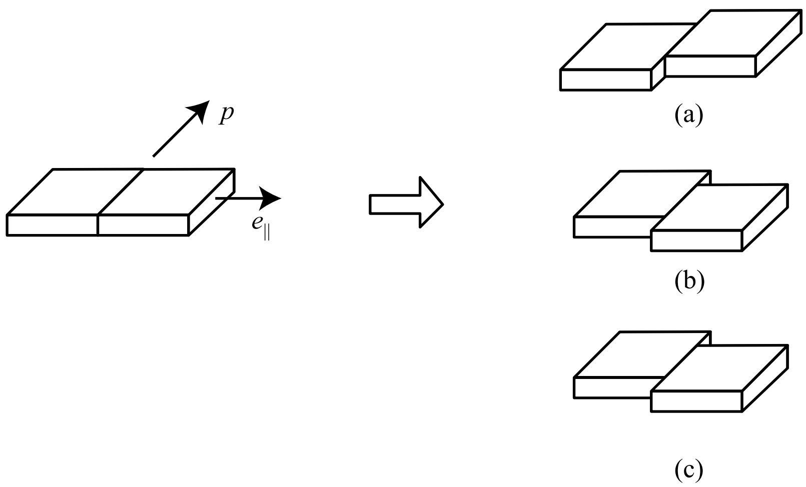
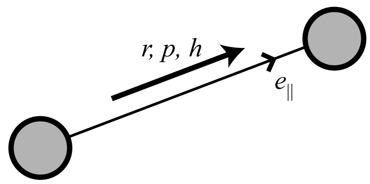


Figure 2: Examples of the character of the freedom associated with a prismatic joint under symmetry operations. (a) A relative displacement between two bodies. (b) The same relative displacement followed by a rotation of the whole system by π about p: $\chi_p = 1$; $\chi_{e_{\|}} = -1$; and, by (14), $\chi_{\text{prismatic}} = 1 \times -1 = -1$. (c) The same original relative displacement followed by a reflection in the plane perpendicular to p: $\chi_p = -1$; $\chi_{e_{\|}} = 1$; and, by (14), $\chi_{\text{prismatic}} = -1 \times 1 = -1$.



(a)

	E	$C_\infty(\phi)$	C_2'	σ_v	σ_h	i	$S_\infty(\phi)$
$\chi_{R_r} = \chi_{R_h}$	1	1	−1	−1	1	1	1
$\chi_{T_p} = \chi_{T_h}$	1	1	−1	1	−1	−1	−1
$\chi_{e_\parallel}$	1	1	−1	1	−1	−1	−1
χ_{revolute}	1	1	1	−1	−1	−1	−1
$\chi_{\text{prismatic}}$	1	1	1	1	1	1	1
χ_{screw}	1	1	1	·	·	·	·

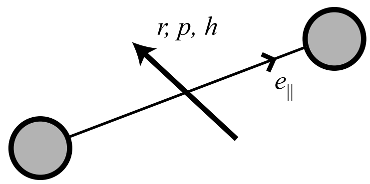
(b)

Figure 3: Local site symmetry $D_{\infty h}$. (a) The disposition of bodies, joint axis r, p, h and edge e; the joint axis will be considered as a pseudo-vector r when the hinge is a revolute joint, and a vector p when it is a prismatic joint; when the hinge is a screw joint, there is no distinction. (b) The characters χ_{revolute}, $\chi_{\text{prismatic}}$, χ_{screw}, $\chi_{R_r} = \chi_{R_h}$, $\chi_{T_p} = \chi_{T_h}$ and $\chi_{e_\parallel}$ for all symmetry operations. Improper operations are not defined for the screw freedom.

and hinge are collinear) or D_{2h} (when the edge and hinge are perpendicular). These two site groups are the maximum achievable; the actual site symmetry may be lower. The two limiting cases are shown in Figs. 3(a) and 4(a), and the tables in Figs. 3(b) and 4(b) show the relevant characters for the possible symmetry operations, which confirm the correctness of (13)–(15).

These formal results have a ready pictorial interpretation. In the $D_{\infty h}$ (or subgroup) configuration, the freedom of a revolute hinge spans the pseudo-scalar symmetry Γ_ϵ, as it is preserved by all proper operations, and reversed by all improper operations; the same behaviour would be found for a cylinder lying along the hinge axis marked with counter-rotating circular arrows at opposite ends. The freedom of a prismatic hinge in the same configuration spans the totally symmetric Γ_0, as in this case it behaves as a simple edge stretch. In the chiral site group appropriate to a screw joint, D_∞ (or subgroup), the freedom of the screw joint spans $\Gamma_\epsilon = \Gamma_0$.

Likewise, in the D_{2h} (or subgroup) configuration, the freedom of a revolute hinge spans the symmetry of a translation normal to the plane containing



(a)

	E	$C_2(p)$	$C_2(e_{\parallel})$	$C_2(p \times e_{\parallel})$	i	$\sigma(p)$	$\sigma(e_{\parallel})$	$\sigma(p \times e_{\parallel})$
$\chi_{R_r} = \chi_{R_h}$	1	1	-1	-1	1	1	-1	-1
$\chi_{T_p} = \chi_{T_h}$	1	1	-1	-1	-1	-1	1	1
$\chi_{e_{\parallel}}$	1	-1	1	-1	-1	1	-1	1
χ_{revolute}	1	-1	-1	1	-1	1	1	-1
$\chi_{\text{prismatic}}$	1	-1	-1	1	1	-1	-1	1
$\chi_{\text{prismatic}}$	1	-1	-1	1	·	·	·	·

(b)

Figure 4: Local site symmetry D_{2h}. (a) The disposition of bodies, joint axis r, p, h and edge e; the joint axis will be considered as a pseudo-vector r when the hinge is a revolute joint, and a vector p when it is a prismatic joint; when the hinge is a screw joint, there is no distinction. (b) The characters χ_{revolute}, $\chi_{\text{prismatic}}$, χ_{screw}, $\chi_{R_r} = \chi_{R_h}$, $\chi_{T_p} = \chi_{T_h}$ and $\chi_{e_{\parallel}}$ for all symmetry operations. Improper operations are not defined for the screw freedom.

the hinge axis and the edge of C. The freedom of a prismatic hinge spans the symmetry of a rotation about this normal. In the chiral site group appropriate to a screw joint, D_2 (or subgroup), the symmetry of the freedom, a translation along the normal, and a rotation about the normal, are all identical.

An alternative to using (13)–(15) is to imagine C as decorated with appropriate motif for each joint freedom: revolute freedoms would be replaced by marked cylinders or simple vectors, prismatic freedoms would be replaced by structureless points or circular arrows, and screws would be replaced by structureless points or vectors. The characters of the freedoms would then correspond to the characters of these motifs. This is analogous to working out vibrational properties of molecules by attaching local triads of displacement vectors to all atoms [14].

It is interesting to note that in both configurations, the character of a revolute freedom is that of a rotation about the axis of the freedom, as long as the bodies connected by the hinge are unshifted (since then $\chi_{e_\parallel}(S) = 1$). Similarly the character of a prismatic freedom is that of a displacement along the axis of the prismatic freedom, as long as the bodies connected by the hinge are unshifted.

3.5.2 Higher dimensional freedoms

We will not deal explicitly here with joints that have more that one degree of freedom (e.g. cylindrical, or spherical joints), as the associated freedoms in these cases can be considered as the sum of one-dimensional freedoms of the types that we have already considered, with appropriate identification of axes.

3.6 Rigid joints

Equation (10) gave an expression for the representation of the constraints imposed for hypothetical rigid joints, but without proof. That proof is given here.

Connection of two bodies through a rigid joint suppresses six relative freedoms, and hence the character $\chi_{\text{rigid}}(S)$ under some symmetry operation S will be given by the sum of characters for the six relative freedoms. But these freedoms can be considered as three relative rotations and three relative translations, and explicit formulae for their characters have been given in (13) and (14). Summing over the freedoms,

$$\chi_{\text{rigid}}(S) = (\chi_T(S) + \chi_R(S))\chi_{e_\parallel}(S), \tag{16}$$

where $\chi_T(S)$ is the character of the set of three mutually perpendicular translations, $\chi_R(S)$ is the character of the set of rotations about three mutually perpendicular axes, and as before, $\chi_{e_\parallel}(S)$ is the character of the inter-body vector e, all under the operation S.

Summing over the joints, or equivalently over the set of inter-body vectors that make up the edges of C, the representation Γ(rigid joints) is seen to be

$$\Gamma(\text{rigid joints}) = \Gamma_\parallel(e, C) \times (\Gamma_T + \Gamma_R) \tag{17}$$

i.e. (10).

4 Example

The use of the symmetry-extended mobility rule will be demonstrated by analysing an overconstrained spatial four-bar mechanism. This is a straightforward pedagogical example: it demonstrates the existence and symmetry of a full-cycle mechanism, as well as the symmetries of states of self stress.

The example linkage that will be analysed here is shown in Fig. 5. In the initial fully-open state it consists of four bars of square cross-section lying in a horizontal plane, connected by revolute joints alternately on the inside and outside top faces. The linkage is in fact a highly symmetric form of a Bennett linkage [1, Chapter 10], and hence must have one degree of freedom. This linkage is particularly interesting, as it folds into a compact bundle of bars, and this gives it potential as the basis for useful deployable structures [15].

Evaluating the scalar extended mobility rule (2) for this structure is uninformative, as the result, $m - s = -2$, is compatible with $m = 0$, $s = 2$. As we shall see, the symmetry rule is more successful in that it predicts the existence of the one mechanism.

A view of the frame, and its contact polyhedron C, in the fully-open state, is shown in Fig. 6. Although the physical embodiment of the frame has symmetry C_{2v}, the contact polyhedron C, when decorated with the four hinge axes, has the higher symmetry $G(C) = D_{2d}$, and it is in this group that we shall initially work. The character table for D_{2d} is shown in Table 1.

The evaluation of the mobility using the new rule, (12), for the example linkage is shown below. The calculation is similar to examples given elsewhere for the symmetry-extended Maxwell rule [5].

The terms of the right-hand side of (12) are evaluated in turn. The permutation representation of the vertices of C has character four under the identity, as there are four vertices, and character two under $S = C_2'$, as

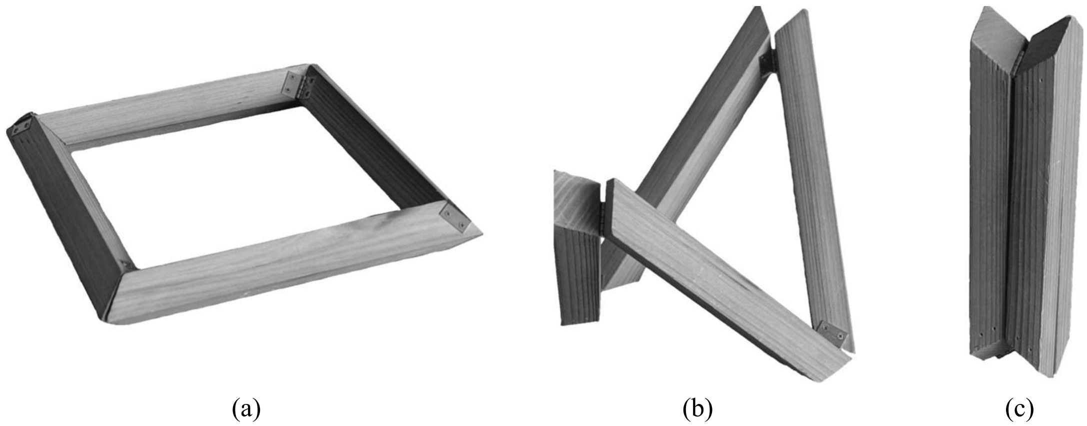


Figure 5: A spatial four-bar linkage, shown folding from its fully open state (a), to its fully closed state (c), where it forms a compact bundle of bars.

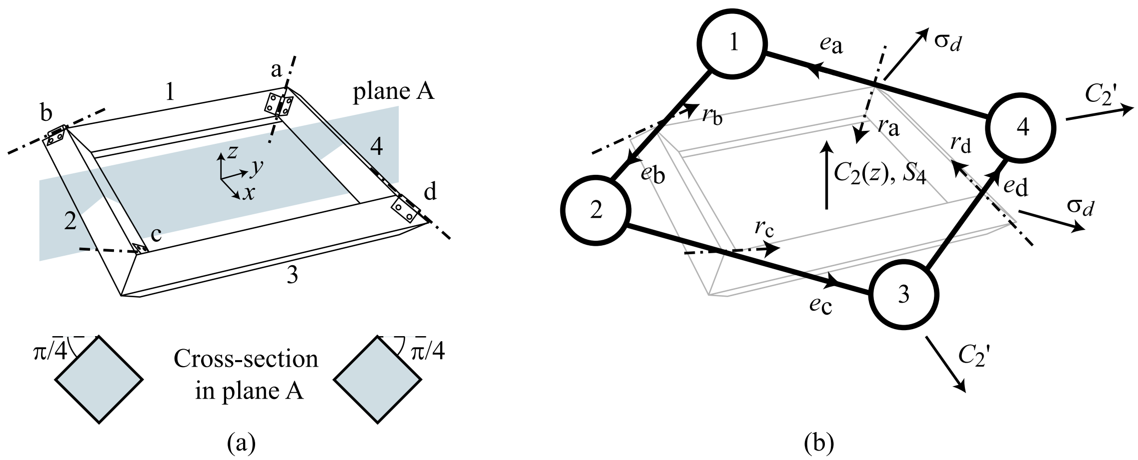


Figure 6: A spatial four-bar linkage. (a) The physical embodiment, showing four bars, 1–4, connected by four revolute joints, a–d. (b) A representation of the contact polyhedron, C, with the four bars shown as rigid bodies, connected by edges across the hinges. The axes of the revolute hinges a–d are shown as pseudo-vectors r_{a}–r_{d}. Symmetry operations in $G(C)$ are shown.

D_{2d}	E	$2S_4$	$C_2(z)$	$2C'_2$	$2\sigma_d$	rotations, displacements
A_1	1	1	1	1	1	
A_2	1	1	1	-1	-1	R_z
B_1	1	-1	1	1	-1	
B_2	1	-1	1	-1	1	z
E	2	0	-2	0	0	(x,y) $(R_x,\ R_y)$

Table 1: The D_{2d} character table.

two vertices lie on each transverse two-fold axis; all other operations shift all vertices. The representation of edge vectors similarly has character four under $S = E$, and character minus two under the dihedral reflections $S = \sigma_d$ which preserves two edge mid-points but reverses the edge vectors. The totally symmetric representation Γ_0 has $\chi(S) =$ for all S. The compound representation $\Gamma_T + \Gamma_R$ may be read from the character table in Table 1. As a sum of translational and rotational terms it has zero character under every improper operation.

The evaluation of Γ_f is the novel feature here. The revolute hinges are left unshifted only by the symmetry operations E and σ_d, and hence the character χ_f under any other symmetry operation is zero. For $S = E$ all four hinges are unshifted, and $\chi_f(E) = 4$. For $S = \sigma_d$, for each hinge $\chi_{R_r}(\sigma_d) = -1$, $\chi_{e_\parallel}(\sigma_d) = -1$, and hence $\chi_{\text{revolute}}(\sigma_d) = -1 \times -1 = 1$. Summing this over the two hinges that are unshifted gives $\chi_f(\sigma_d) = 2$. In tabular form, the full calculation is

D_{2d}	E	$2S_4$	$C_2(z)$	$2C'_2$	$2\sigma_d$
$\Gamma(v, C)$	4	0	0	2	0
$-\Gamma_\parallel(e, C)$	-4	0	0	0	2
$-\Gamma_0$	-1	-1	-1	-1	-1
$=$	-1	-1	-1	1	1
$\times(\Gamma_T + \Gamma_R)$	6	0	-2	-2	0
$=$	-6	0	2	-2	0
$+\Gamma_f$	4	0	0	0	2
$= \Gamma(m) - \Gamma(s)$	-2	0	2	-2	2

Evaluating $\Gamma(m) - \Gamma(s)$ in terms of irreducible representations gives

$$\Gamma(m) - \Gamma(s) = -B_1 + B_2 - E, \tag{18}$$

and hence

$$\Gamma(m) \supset B_2 \tag{19}$$

and

$$\Gamma(s) \supset B_1 + E. \tag{20}$$

The structure has at least one mechanism, with fully defined symmetry. The rule has found the mechanism that we could not detect from simple counting.

Just as the algebraic rule gives only the excess $m - s$, and cannot fix m and s separately, symmetry arguments give only the symmetry excess $\Gamma(m) - \Gamma(s)$. It is always conceivable that there may exist cancelling equisymmetric mechanisms and states of self-stress. In fact, in the present example, there are no more mechanisms to be found; this can be checked by more detailed numerical calculations or experiments.

In favourable cases, symmetry not only reveals the existence of a mechanism, but can show that it must have full-cycle mobility [16]: if in some symmetry group, a mechanism is fully symmetric, and there is no equisymmetric state of self-stress, that mechanism must have full-cycle mobility (must be finite).

The calculation tabulated above shows that there is a single instantaneous mobility with less than full symmetry in $G(C)$. It is clear that mobilising the linkage will remove some, but not all, of its symmetries. Displacement of the linkage along its B_2 path reduces the symmetry of C from D_{2d} to C_{2v}. Re-evaluating $\Gamma(m) - \Gamma(s)$ in the lower symmetry group, C_{2v}, either from scratch, or by deleting columns of the calculation table, gives

$$\Gamma(m) - \Gamma(s) = A_1 - A_2 - B_1 - B_2, \tag{21}$$

and because we know, from numerical calculation or otherwise, that only one mechanism exists, we can assert that $\Gamma(m) = A_1$, and $\Gamma(s) = A_2 + B_1 + B_2$, and hence that the mechanism has full-cycle mobility. A general set of tables for the symmetry-aided detection of finite mechanisms is given in [17].

5 Conclusion

The conventional mobility criterion for a linkage has been subsumed into an extended form that considers the symmetries, as well as the numbers, of mobilities and states of self stress. It has been stated in a compact general form, with explicit formulae in terms of one-degree-of-freedom lower pairs. A pedagogical example demonstrates the use of the rule in detecting and classifying mobility, and further its role in determining whether the mobility is full-cycle.

The full-cycle mobility of the example four-bar linkage, shown here using the symmetry-extended mobility rule, will come as no surprise to those familiar with overconstrained mechanisms. However, the strength of the new rule is that it can be applied to much more complex systems; a good example is the *expandohedron*, which models a possible mechanism for the expansion of an icosahedral virus, and is analysed by the new rule in [18]. Symmetry proves to be essential to the understanding of the deployment mode of these expanding polyhedra.

Whenever a linkage has configurations of non-trivial symmetry, the symmetry-extended mobility rule can give more specific information on mobility and states of self stress than the traditional scalar mobility criterion. The new rule is easy to apply, requiring only consultation of a character table and counting of structural components shifted and unshifted by symmetry operations.

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