Symmetry conditions and finite mechanisms

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July 13, 2006

Abstract

Using group representation theory, a simplified criterion for the detection of finite symmetric mechanisms is presented.

1 Introduction

The identification of finite mechanisms in statically and kinematically indeterminate structures is, in general, a difficult problem. However, Kangwai and Guest (1999) showed that in certain cases finiteness of mechanisms could be found using only symmetry arguments and a linear analysis. Here we revisit Kangwai and Guest's method to show that their symmetry arguments can be straightforwardly stated in terms of representations of mechanisms and states of self-stress in the point group of the structure, giving an immediate assessment of the finiteness of mechanisms for many cases.

For any kinematically indeterminate structure, it is possible to find a set of mechanisms, i.e., displacements which to first order cause no deformation of structural elements. (Here it is usual to exclude rigid body motions.) Mechanisms may be either *finite*, in which case there is a continuous displacement path that is compatible at every point with zero deformation of the structure, or *infinitesimal*, in which case there is deformation at second or higher order. Determination of the finite nature of a mechanism in general requires non-linear analysis (Tarnai, 1989; Calladine and Pellegrino, 1992;

Salerno, 1992; Connelly and Servatius, 1994; Tarnai and Szabó, 2000; Garcea *et al.*, 2005). Kuznetsov (2000) has stressed the difficulties that may arise with 'singular' (e.g., highly symmetric) configurations, but nonetheless, the behaviour at points of high symmetry is often a useful guide to that of physical systems, where the symmetry may be only approximate. Kangwai and Guest (1999) introduced, for specific symmetric cases, a criterion that could determine the finiteness of a mechanism based on purely first-order analysis combined with a symmetry argument, and has proved to be applicable to a wide variety of structures (Kovács et al., 2004; Fowler and Guest, 2005). We show here that there is a simple and general way of determining finiteness according to this criterion, obviating the need for explicit calculation in every particular case.

The difficult cases for determining finiteness of mechanisms are those where structures are also statically indeterminate, and hence have states of self-stress, i.e., sets of internal stresses in self-equilibrium in the absence of externally applied loads. The symmetry finiteness criterion, as stated by Kangwai and Guest (1999) is as follows.

Proposition 1 *If a mechanism is fully-symmetric in some subgroup of the symmetry group of the structure, with no equisymmetric state of self-stress, then that mechanism must be finite.*

However, the converse does not always hold: if such an equisymmetric state of self-stress exists, then the mechanism *may* be stiffened, and hence be only infinitesimal, or *may* still be finite. A celebrated example where the converse of the proposition would not apply is the Connelly-Servatius (1994) cusp mechanism.

The present paper reformulates the symmetry finiteness criterion in a way that avoids the need to consider sub-groups of the symmetry group of the structure. Statement and proof of the new formulation in cases where there is a mechanism belonging to a non-degenerate representation follows in Section 2. This covers all cases that have been analysed so far with the symmetry finiteness criterion. For completeness, the present paper briefly considers, in Section 3, the consequences of degeneracy. Section 4 contains a number of examples of the criterion.

2 A symmetry finiteness criterion based on representations

For mechanisms that belong to a non-degenerate representation, it can be shown that the following proposition is equivalent to the symmetry criterion stated by Kangwai and Guest.

Proposition 2 *A mechanism that belongs to a non-degenerate representation will be finite if, in the point group of the undisplaced object, there is neither a state of self-stress that is equisymmetric with the mechanism, nor a totally symmetric state of self-stress.*

The proposition can be proved as follows.

Suppose that a structure has a configuration with point-group symmetry G, and in that configuration has mechanisms spanning the (reducible) representation $\Gamma(m)$ of G, and states of self-stress spanning the representation $\Gamma(s)$. We will initially concentrate on one member of the set of mechanisms, m_1, a mechanism with non-degenerate, irreducible representation Γ_{m_1}. We can assume that the mechanism is not totally symmetric, as if it were, Proposition 1 would apply directly: a totally symmetric mechanism will be finite if there is no equisymmetric state of self-stress. Displacement of the structure along m_1 gives a new configuration with point group symmetry H_1; H_1 is a subgroup of G defined entirely by Γ_{m_1}. (Using notation that will be defined in Section 3, H_1 is the *kernel* of G under Γ_{m_1}.)

Let G consist of symmetry operations $R_i, i = 1 \ldots |G|$ and let the characters of Γ_{m_1} be $\chi_{m_1}(R_i)$. Then H_1 is a subgroup of G, of order $|H_1| = |G|/2$, comprising those operations R_i of G for which $\chi_{m_1}(R_i) = +1$. It is easy to see that this condition on the characters defines a group. As the characters of a non-degenerate irreducible representation obey the group multiplication table, i.e., $\chi(R_i)\chi(R_j) = \chi(R_k)$ for $R_iR_j = R_k$, the set of operations with character $+1$ is closed under multiplication, includes the identity, contains an inverse for every operation in the set, and inherits the associative property from G.

Suppose that $\Gamma(s)$ is not empty, and consider a state of self-stress with irreducible representation Γ_s, say, as a candidate for 'blocking' Γ_{m_1}, i.e., stiffening the mechanism m_1. There are three possibilities:

(i) Γ_s is the totally symmetric representation, Γ_0, in G;

(ii) Γ_s is Γ_{m_1} in G;

(iii) Γ_s is neither Γ_0 nor Γ_{m_1} in G.

As a non-degenerate and non-totally symmetric irreducible representation, Γ_{m_1} has character $+1$ for exactly half of the operations R_i of G, and character -1 for the other half (by orthogonality with Γ_0). For convenience, we will choose an ordering of the operations such that $\chi_{m_1}(R_i) = +1$ for

$i = 1, \ldots, |G|/2$, and $\chi_{m_1}(R_i) = -1$ for $i = |G|/2 + 1, \ldots, |G|$. With this ordering, let the characters of the representation of the state of self-stress, Γ_s, be $\chi_s(R_i) = \alpha_i$ and $\chi_s(R_{(|G|/2+i)}) = \beta_i$ for $i = 1, \ldots, |G|/2$ with

$$\alpha = \sum_{i=1}^{|G|/2} \alpha_i \quad ; \quad \beta = \sum_{i=1}^{|G|/2} \beta_i.$$

The various characters are summarized in the table below.

G	R_1	$\cdots$	$R_{\|G\|/2}$	$R_{\|G\|/2+1}$	$\cdots$	$R_{\|G\|}$
Γ_0	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$
Γ_{m_1}	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1
Γ_s	α_1	$\cdots$	$\alpha_{\|G\|/2}$	β_1	$\cdots$	$\beta_{\|G\|/2}$

In case (i), we have $\alpha_i = \beta_i = +1$, and $\Gamma_s = \Gamma_0$ in both G and H_1. In case (ii), $\alpha_i = -\beta_i = +1$, and $\Gamma_s = \Gamma_0$ in H_1, but not G. In case (iii), orthogonality of Γ_s to Γ_0 gives

$$\alpha + \beta = 0,$$

and orthogonality to Γ_{m_1} gives

$$\alpha - \beta = 0,$$

and hence $\alpha = \beta = 0$; $\alpha = 0$ implies that Γ_s remains orthogonal to Γ_0 (and hence to Γ_{m_1}) in H_1. Thus in case (i) state of self-stress s may block mechanism m_1 in both G and H_1; in case (ii) s may block m_1 in H_1; in case (iii) s does not block m_1. Notice that the above applies equally to degenerate and non-degenerate Γ_s. Case-by-case consideration has therefore shown the truth of Proposition 2.

Details of the identification of Γ_{m_1} and its associated group H_1 can be filled in from standard character and descent in symmetry tables (e.g. Atkins, Child and Phillips (1970); Salthouse and Ware (1972); Altmann and Herzig (1994)).

So far we have considered a single non-degenerate mechanism. If the configuration that has G symmetry allows several such mechanisms, but displacement occurs along only one of them, the above reasoning applies directly. If, instead, displacement is along some linear combination of such mechanisms, the consequences are easily worked out. For example, suppose that we have mechanisms m_1 and m_2 of distinct symmetries in G, Γ_{m_1} and Γ_{m_2}. Displacement along a linear combination of m_1 and m_2 can be analysed with the help of the character table below, where the operations of G have been separated into equal-sized blocks according to their characters for the irreducible representations Γ_{m_1} and Γ_{m_2}.

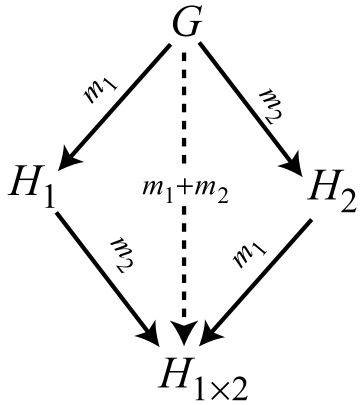

Figure 1: The descent in symmetry from G arising from displacement along mechanisms m_1 and m_2, alone and in combination.

G	R_1	$\cdots$	$R_{\lvert G\rvert/4}$	R'_1	$\cdots$	$R'_{\lvert G\rvert/4}$	R''_1	$\cdots$	$R''_{\lvert G\rvert/4}$	R'''_1	$\cdots$	$R'''_{\lvert G\rvert/4}$
Γ_0	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$
Γ_{m_1}	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1	$\cdots$	-1
Γ_{m_2}	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1
$\Gamma_{m_1} \times \Gamma_{m_2}$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$
Γ_s	α_1	$\cdots$	$\alpha_{\lvert G\rvert/4}$	β_1	$\cdots$	$\beta_{\lvert G\rvert/4}$	γ_1	$\cdots$	$\gamma_{\lvert G\rvert/4}$	δ_1	$\cdots$	$\delta_{\lvert G\rvert/4}$

The operations $\{R_1 \dots R_{|G|/4}\} + \{R'_1 \dots R'_{|G|/4}\}$ constitute the group H_1 which is reached from G by a pure m_1 distortion. Similarly the group H_2 reached from G by a pure m_2 distortion consists of the R and R'' operations. The R operations by themselves define the group $H_{1\times2}$, which is reached from G by a displacement along a generic combination of m_1 and m_2. The relationships between the various subgroups of G are shown schematically in Figure 1. By definition, Γ_{m_1} and Γ_0 become totally symmetric in H_1, and Γ_{m_2} and Γ_0 become totally symmetric in H_2. In the group $H_{1\times2}$, Γ_{m_1}, Γ_{m_2}, $\Gamma_{m_1} \times \Gamma_{m_2}$ and Γ_0 become totally symmetric. Now consider a candidate state of self-stress, s. Its characters are defined in the table, and we define the partial sums

$$\alpha = \sum_{i=1}^{|G|/4} \alpha_i \quad ; \quad \beta = \sum_{i=1}^{|G|/4} \beta_i \quad ; \quad \gamma = \sum_{i=1}^{|G|/4} \gamma_i \quad ; \quad \delta = \sum_{i=1}^{|G|/4} \delta_i .$$

There are five possibilities for Γ_s:

(i) Γ_s is the totally symmetric representation, Γ_0, in G: $\alpha_i = \beta_i = \gamma_i = \delta_i = +1$, and $\Gamma_s = \Gamma_0$ in G and all subgroups. Thus state of self-stress s may block mechanism m_1 and m_2 in any combination.

(ii) Γ_s is Γ_{m_1} in G: $\alpha_i = \beta_i = -\gamma_i = -\delta_i = +1$, and $\Gamma_s = \Gamma_0$ in H_1 and $H_{1\times2}$, but not H_2. Thus, s may block all but pure m_2.

(iii) Γ_s is Γ_{m_2} in G: $\alpha_i = -\beta_i = \gamma_i = -\delta_i = +1$, and $\Gamma_s = \Gamma_0$ in H_2 and $H_{1\times2}$, but not H_1. Thus, s may block all but pure m_1;

(iv) Γ_s is $\Gamma_{m_1} \times \Gamma_{m_2}$ in G: $\alpha_i = -\beta_i = -\gamma_i = \delta_i = +1$, and $\Gamma_s = \Gamma_0$ in $H_{1\times2}$, but not H_1 or H_2. Thus s may block all but pure m_1 or pure m_2.

(v) Γ_s is none of the above. Orthogonality gives:

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 0\\
\alpha + \beta - \gamma - \delta &= 0\\
\alpha - \beta + \gamma - \delta &= 0\\
\alpha - \beta - \gamma + \delta &= 0
\end{aligned}$$

and hence $\alpha = 0$, implying that Γ_s remains orthogonal to Γ_0 in H_1, H_2 and $H_{1\times2}$. Hence, s does not block m_1, m_2, or any combination of m_1 and m_2.

This reasoning can be extended to apply Proposition 2 to any combination of non-degenerate mechanisms.

3 Mechanisms described by degenerate representations

When a mechanism is d-fold degenerate, the symmetry possibilities for distortion and blocking by states of self-stress are more involved, as the system can visit different subgroups of G by following different combinations of the d components of the mechanism. An established notation for the relations between the various groups is used, for example, in vibrational spectroscopy (McDowell, 1965), and can be used to frame some general remarks on how degenerate and non-degenerate mechanisms are blocked.

Let the irreducible representation of the mechanism be the d-fold degenerate Γ_{md}. The lowest symmetry group, reached by a generic combination of the d components of the mechanism, is the *kernel* of Γ_{md}. The kernel is an invariant subgroup of G and consists simply of those elements of G whose characters for Γ_{md} are equal to d. For any degenerate representation, the kernel is easily identified from the character table. In the kernel, Γ_{md} reduces to d copies of Γ_0. In the present context, it can be seen that, if no state of self-stress becomes totally symmetric in the kernel, then all combinations of the d components of m_d are finite mechanisms. Given that the kernel is not necessarily equal to the trivial group C_1, it is possible therefore for a system to support a number of states of self-stress that cannot block a given degenerate finite mechanism.

Unlike the non-degenerate case, the symmetries accessible to a degenerate mechanism are not necessarily restricted to the kernel group. By particular choices of combination, it may be possible to retain symmetry elements additional to those in the kernel, and thus produce configurations belonging to point groups of which the kernel is a subgroup. The accessible groups are the *cokernels* of Γ_{md}; McDowell (1965) discusses the identification of cokernels, and lists them for the degenerate representations of a number of spectroscopically important point-groups.

The existence of cokernels for some degenerate representations widens the scope for finite degenerate mechanisms. Even in cases where the generic mechanism is blocked in the kernel, there may be combinations of the d components that access a cokernel in which no state of self-stress is totally symmetric, and by Proposition 1, those specific combinations will remain finite.

As an example, consider a hypothetical system of D_{6h} symmetry where $\Gamma(m) = E_{2g}$ and $\Gamma(s) = A_{2g}$. The relevant rows of the D_{6h} character table are shown below.

D_{6h}	E	$2C_6$	$2C_3$	C_2	$3C_2'$	$3C_2''$	i	$2S_3$	$2S_6$	σ_h	$3\sigma_d$	$3\sigma_v$
A_{2g}	1	1	1	1	-1	-1	1	1	1	1	-1	-1
E_{2g}	2	-1	-1	2	0	0	2	-1	-1	2	0	0

McDowell gives the kernel of E_{2g} as C_{2h}, and this can be confirmed by inspection of the table above, as the four columns with character $+2$ are those for E, C_2, i and σ_h. It can also be seen by inspection that $\Gamma(s) = A_{2g}$ becomes totally symmetric in C_{2h} and hence we cannot state that the pair of mechanisms is finite. However, the cokernel of E_{2g} is D_{2h} (McDowell, 1965), and as the table shows, A_{2g} is not totally symmetric in D_{2h} (four characters are $+1$, four characters are -1 under these operations). Therefore it is guaranteed that the combination of components that lead from D_{6h} to D_{2h} is a finite mechanism.

4 Examples

4.1 Structure stiffened by self-stress

Figure 2 shows a planar pin-jointed framework that has been analysed by Kangwai and Guest (2000). Considered in two dimensions, a structure with this connectivity is generically both statically and kinematically determinate, but in the configuration shown has one state of self-stress and one mechanism. The planar structure has point group C_{3v}, with

$$\Gamma(m) = A_2,\ \Gamma(s) = A_1.$$

The single mechanism has the symmetry of an in-plane rotation of a central triangle, and the state of self-stress corresponds to a totally symmetric distribution of tensions in the bars. As the single state of self-stress is totally symmetric in C_{3v}, it can in principle stiffen any mechanism, and inspection, or a formal analysis of the tangent stiffness (see, e.g., Guest, 2005) shows that the mechanism is indeed stiffened.

We can also consider a structure in three dimensions that has the same set of connections. In a generic configuration, such a structure has three mechanisms, and no state of self-stress. Clearly these mechanisms must be finite. However, in the particular planar configuration shown, the structure attains D_{3h} symmetry, where it has a single state of self-stress and four mechanisms. The symmetry form of the Maxwell rule for pin-jointed frameworks (Fowler and Guest, 2000) gives a full account, and yields

$$\Gamma(m) - \Gamma(s) = A'_2 + A''_2 + E''_2 - A'_1.$$

As, by inspection, $\Gamma(s) = A'_1 \equiv \Gamma_0$, the four mechanisms span

$$\Gamma(m) = A'_2 + A''_2 + E''_2.$$

The state of self-stress is fully symmetric in this configuration, and hence can stiffen all mechanisms; analysis of the tangent stiffness shows that this stiffening is effective for all four mechanisms.

4.2 Prestressable finite mechanism

Figure 3 shows a classic example of a type of pin-jointed structure (Tarnai, 1980) that satisfies Maxwell's rule for pin-jointed frames (Calladine, 1978), but nevertheless admits a finite mechanism. The structure shown has a hexagonal ring of bars, connected in triangulated fashion to a rigid base. Its

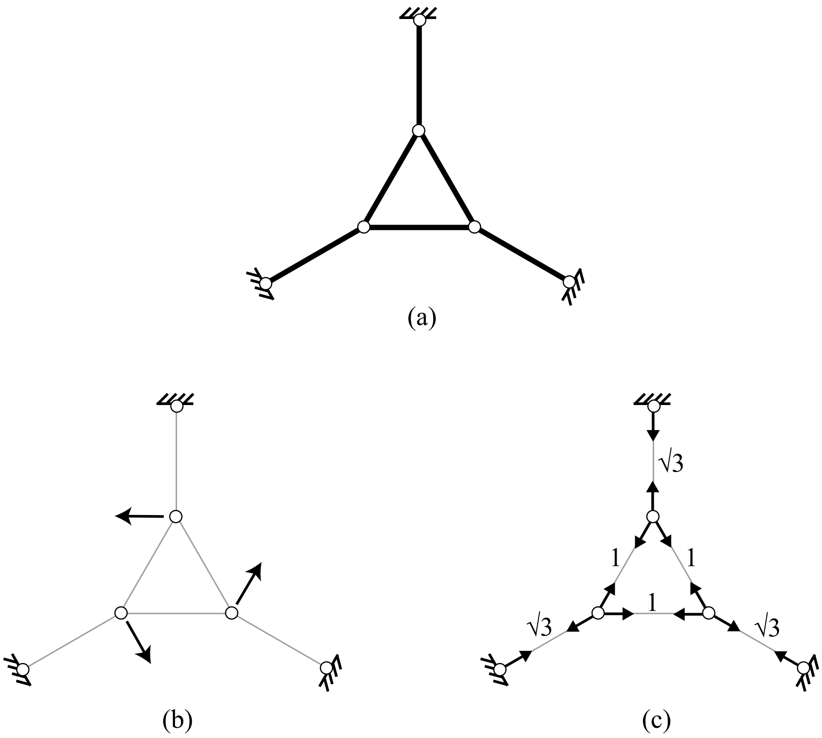

Figure 2: (a) A planar structure in which all mechanisms are stiffened by a state of self-stress; (b) the mechanism, showing directions of infinitesimal nodal displacement; (c) the state of self-stress, showing relative bar tensions.

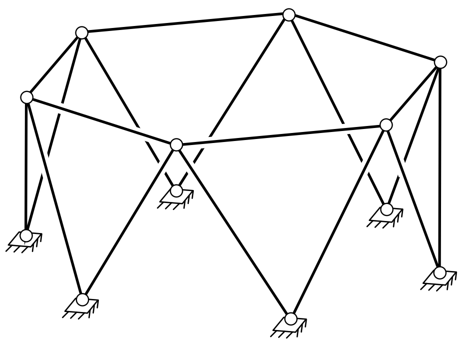

Figure 3: A ring structure with a finite mechanism.

point group is C_{3v}, and as Kangwai and Guest (1999) have shown, the single mechanism has symmetry

$$\Gamma(m) = B_1$$

and the single state of self-stress has

$$\Gamma(s) = B_2.$$

It follows immediately from Proposition 2 that the mechanism is finite: there is neither an equisymmetric nor a totally symmetric state of self-stress here. The B_1 mechanism leads to C_{3v} configurations where the state of self-stress has A_2 symmetry.

Following the finite mechanism eventually takes the structure to an interesting point of kinematic bifurcation, where the hexagon has degenerated into a triangle, as shown in Figure 4. At this point, a new pair of states of self-stress spanning the E representation emerges (Kangwai and Guest, 1999), and hence $\Gamma(m)$ becomes

$$\Gamma(m) = A_1 + E$$

with

$$\Gamma(s) = A_2 + E.$$

The new states of self-stress do not affect the conclusion that there must be a finite A_1 mechanism leading out of this configuration. However, we cannot deduce the existence of further finite mechanisms: the new states of self-stress are equisymmetric with the new mechanisms, and hence could stiffen generic combinations. In fact, in this case, there are three additional finite paths leading away from the bifurcation point, each of which retains C_s symmetry about one of the σ_v reflection planes of the C_{3v} geometry (Kumar

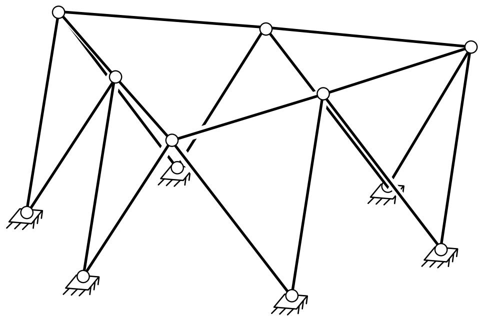

Figure 4: The ring structure shown in Figure 3 displaced along the mechanism path until a point of kinematic bifurcation has been reached.

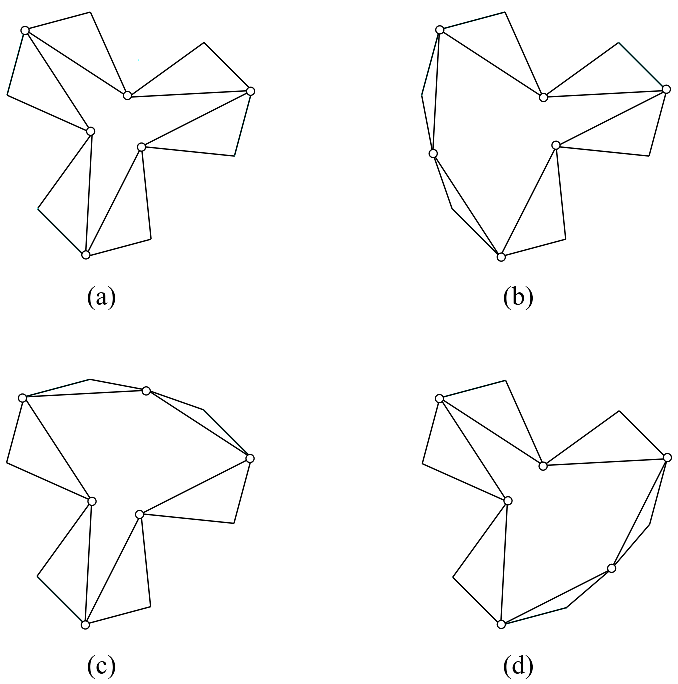

Figure 5: A plan view showing the finite paths leading out of the point of kinematic bifurcation shown in Figure 4; non-foundation joints are shown with a ring, foundation joints without a ring. The displaced structure in (a) retains C_{3v} symmetry, while those shown in (b), (c) and (d) each have C_s symmetry about one of the σ_v reflection planes of the C_{3v} geometry.

and Pellegrino, 2000). C_s is the cokernel of E in C_{3v}, whereas the kernel is the trivial group C_1. The paths are shown in Figure 5. Symmetry analysis shows only that stiffening of the mechanism is predicted, but not that it must occur. As always, symmetry is most powerful when showing that a phenomenon is forbidden, and hence detecting here when mechanisms *must* be finite, as blocking is not allowed, rather than when they *may* be infinitesimal, as blocking is permitted.

Acknowledgements

SDG acknowledges the support of the Leverhulme Trust, PWF the support of the Royal Society through the RS/Wolfson Research Merit Award Scheme.

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