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contents

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TWO-ORBIT SWITCH-PITCH STRUCTURES

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ABSTRACT

The Hoberman 'switch-pitch' ball is a transformable structure with a single folding and unfolding path. The underlying cubic structure has a novel mechanism that retains tetrahedral symmetry during folding. Here, we propose a generalized class of structures of a similar type that retain their full symmetry during folding. The key idea is that we require two orbits of nodes for the structure: within each orbit, any node can be copied to any other node by a symmetry operation. Each member is connected to two nodes, which may be in different orbits, by revolute joints. We will describe the symmetry analysis that reveals the symmetry of the internal mechanism modes for a switch-pitch structure. To follow the complete folding path of the structure, a nonlinear iterative predictor-corrector algorithm based on the Newton method is adopted. First, a simple tetrahedral example of the class of two-orbit structures is presented. Typical configurations along the folding path are shown. Larger members of the class of structures are also presented, all with cubic symmetry. These switch-pitch structures could have useful applications as deployable structures.

Keywords: *Symmetry, Mechanism, Revolute hinge, Compatibility matrix, Folding*

1. INTRODUCTION

Transformable structures or mechanisms can change their geometric configurations without inducing any incompatible deformations of their members. They have potential applications as deployable structures. The 'switch-pitch' ball, invented by Hoberman [1], is a typical transformable structure. It is a novel mechanism with a single folding and unfolding path. In the kinematic analysis, the gears that ensure links rotate together can be neglected, provided singularities in the mechanism are properly avoided [2]. In the previous study [3], we have investigated the Hoberman 'switch-pitch' ball using the symmetry method, and found that the underlying cubic structure can retain tetrahedral symmetry during folding.

In this paper, we will propose a generalized class of switch-pitch structures. The basic design of the structure is that it consists of *nodes*, which are rigid, and are connected to a number of members through revolute joints. Each member connects two nodes. Each revolute joints has an axis that is perpendicular both to the direction of the member, and to a radial vector from the centre of the structure to the centre of the member. A typical node and member is shown in Figure 1. The node N is connected to three members via intersecting revolute joints, and every member has two revolute joints (See the member k in Figure 1(b)). A local right-handed coordinate system is defined for every member of the two-orbit switch-pitch structures: the x direction is along the axis of the member, and the z direction is parallel to the axis of the revolute joints [3].

A key feature of the class of structures considered here is that they each have two *orbits* of nodes. An orbit of nodes is a set of nodes that are equivalent under symmetry, that is, each node within an orbit can be transformed to any other node within the orbit by a symmetry of the structure (for instance, a

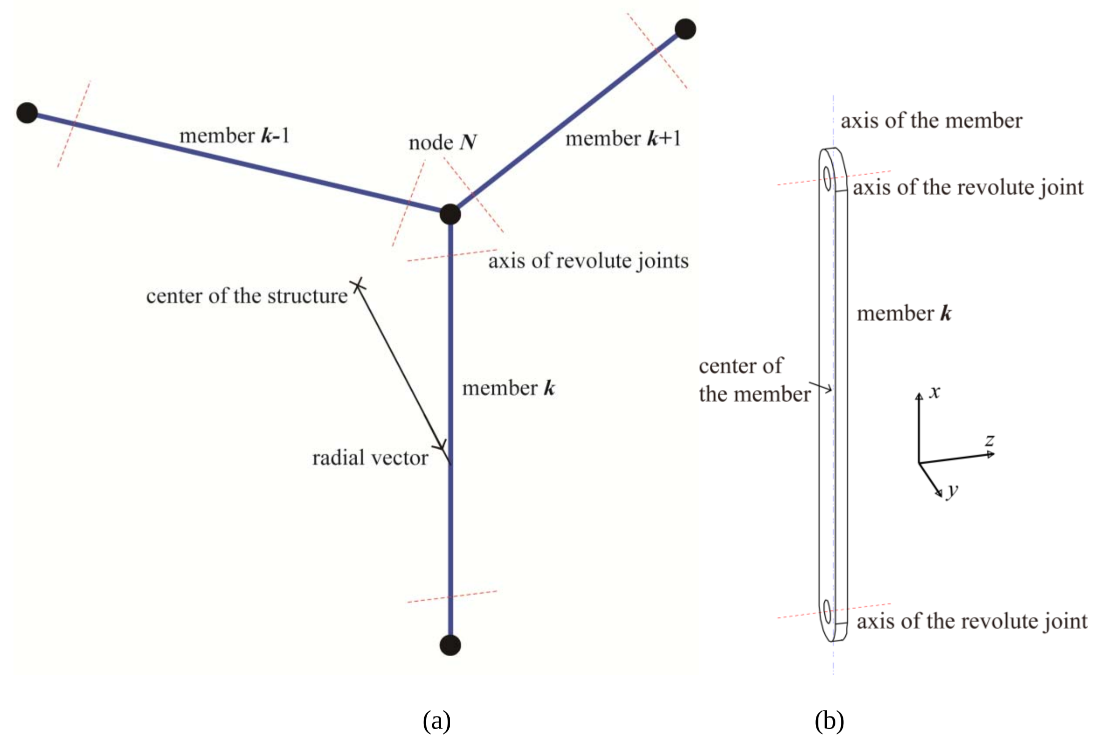


Figure 1. *A typical node N connected to members by revolute joints (a), where each member is associated with two revolute joints and has the local coordinate system shown in (b)*

rotation, or a reflection). In the following sections, we will adopt a symmetry-adapted mobility analysis, to reveal the character of the mechanisms. In addition, a nonlinear iterative predictor-corrector algorithm will be presented, and used to follow the complete folding path of a structure that exhibits its full symmetry.

2. SYMMETRY-ADAPTED MOBILITY ANALYSIS

It is assumed herein that a two-orbit switch-pitch structure consists of n nodes and b members, unrestrained in three dimensions. By counting components, we can find the relative mobility, given by

$$m - s = 6(n - b - 1) + \sum_{i=1}^{b} f_i \qquad (1)$$

where m is the number of independent mechanisms, s is the number of states of self-stress, and f_i is the number of relative freedoms permitted by member i. However, for the switch-pitch structures, counting is not sufficient to reveal that the structure can move, whereas symmetry can reveal more. On the basis of group theory [4] and its matrix representation, Guest and Fowler [5] proposed a symmetry-extended mobility rule for investigating the relative mobility of an over-constrained structure:

$$\Gamma(m) - \Gamma(s) = (\Gamma_T + \Gamma_R) \times \left[\Gamma(v) - \Gamma_{\parallel}(e) - \Gamma_0\right] + \Gamma_f \qquad (2)$$

where $\Gamma(m)$ is the (reducible) representation of the states of independent mechanism, and $\Gamma(s)$ is the representation of the states of self-stress; $\Gamma(v)$ is the (reducible) representation of nodes, with character defined by the numbers of nodes unshifted by each symmetry operation, and $\Gamma_{\parallel}(e)$ is the representation of a set of vectors along the members. Γ_T, Γ_R, and Γ_f are the representations of rigid-body translations, rigid-body rotations, and permitted freedoms, respectively; Γ_0 is the totally symmetric representation. Note that the representations Γ_0, Γ_T, and Γ_R featuring in equation (2) can be found in reference sets of character tables for point symmetry groups, e.g., the book of Altmann and Herzig [4].

After completing the evaluations of $\Gamma(v)$, $\Gamma_{\parallel}(e)$, and Γ_f (i.e., the characters associated with each symmetry operation), we can decompose $\Gamma(m) - \Gamma(s)$ into combinations of a set of irreducible representations:

$$\Gamma(m)-\Gamma(s)=\sum_j \alpha_j \Gamma_j \qquad (3)$$

where α_j is the weighting coefficient for the *j*th irreducible representation Γ_j of the symmetry group. In fact, $\Gamma(m)$ and $\Gamma(s)$ can be expressed independently through the sign of each coefficient α_j, since both $\Gamma(m)$ and $\Gamma(s)$ contain only non-negative numbers of irreducible representations. Through the irreducible representations contained in $\Gamma(m)$, we can reveal the symmetry of the internal mechanism modes. Guest and Fowler [6] have proved that if a mechanism is fully symmetric in the symmetry group of the structure, with no counterbalancing equisymmetric state of self-stress, then the mechanism must be finite. This symmetry condition for mechanisms can directly determine whether the structure retains its full symmetry during the motion.

3. ALGORITHM FOR FOLLOWING THE COMPLETEFOLDING PATH

As shown in Figure 1(b), since the revolute joints are attached at the two ends, each member permits two relative freedoms, i.e., the relative shear motion along the local y direction, and the relative bend around the local z direction. The full compatibility equation for a switch-pitch structure can be written as

$$C_{4b\times 6n}\cdot d_{6n\times 1}=e_{4b\times 1} \qquad (4)$$

where C is the compatibility matrix, containing geometric information (coordinates and orientations in the global right-handed coordinate system); d is the displacement vector, describing infinitesimal movements of nodes; and e is the deformation vector, expressing four constraints for each member. The number of independent mechanisms can be obtained through equation (4):

$$m=6n-r-6 \qquad (5)$$

where r is the rank of the compatibility matrix C in equation (4). Note that the six rigid-body modes have been separated from the internal mechanisms, because the switch-pitch structures investigated in this study are unconstrained.

We will confirm below that structures of this type have a finite mechanism. To follow the folding path, a nonlinear iterative predictor-corrector algorithm based on Newton's method is used. The algorithm is summarized by the flowchart in Figure 2.

In the t^{th} step, the structure is displaced by a predicted displacement d^t, which is found from the previous step as the null-space of C^{t-1}, i.e., a

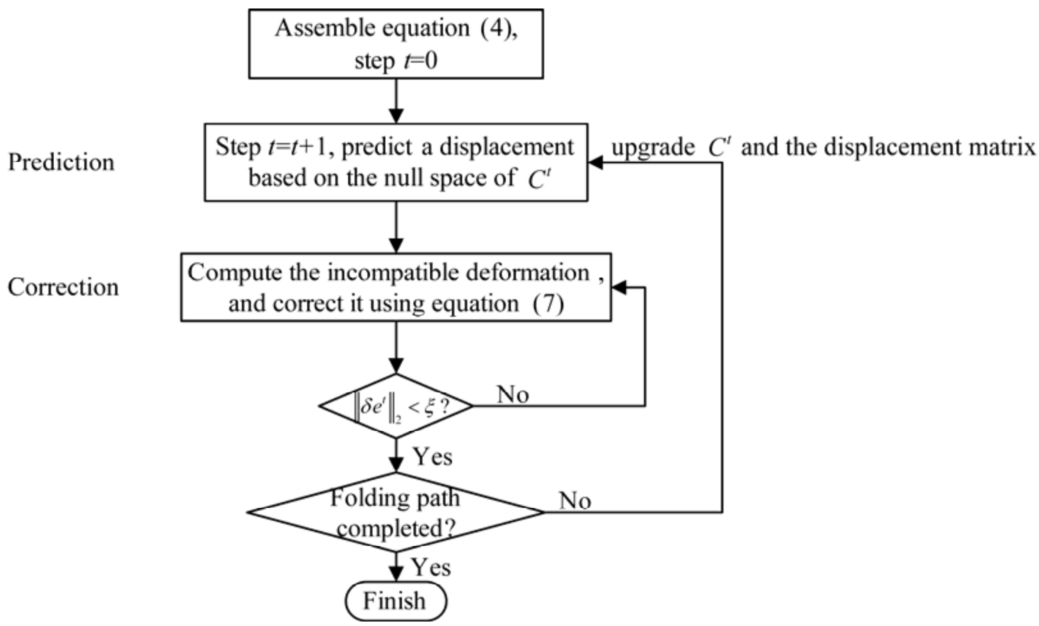


Figure 2. *Flowchart of the nonlinear iterative predictor-corrector algorithm*

solution of $C^{t-1} \cdot d^t = 0$. However, since the folding path is nonlinear, deformation of the members is induced after upgrading the locations of the nodes of the structure; this deformation is then written as a deformation vector:

$$\delta e^t = [\delta l_1, \delta\tau_1, \theta_{1x}, \theta_{1y}, \ldots, \delta l_k, \delta\tau_k, \theta_{kx}, \theta_{ky}, \ldots]^T \quad (6)$$

where, for any member k, δl_k is the axial deformation, $\delta\tau_k$ is the shear deformation (along the z direction), θ_{kx} is the torsion angle, and θ_{ky} is the bending angle (along the local y direction). To correct this deformation, the minimum length least squares solution for the displacement can be computed, using equation (4), as

$$\delta d^t = -(C^t)^+ \cdot \delta e^t \quad (7)$$

where $(C^t)^+$ is the generalized inverse matrix of the compatibility matrix C^t. The deformation will be corrected iteratively, unless the 2-norm of δe^t is less than the allowable error of ξ. Then the location and the compatibility matrix of the structure are upgraded, step by step.

4. EXAMPLES OF THE TWO-ORBIT SWITCH-PITCH STRUCTURES

In this section, we will show some examples of two-orbit switch-pitch structures with high symmetry.

4.1 Switch-Pitch Structure with Point Group Symmetry T_d

As shown in Figure 3, this switch-pitch structure consists of 16 nodes and 24 members.

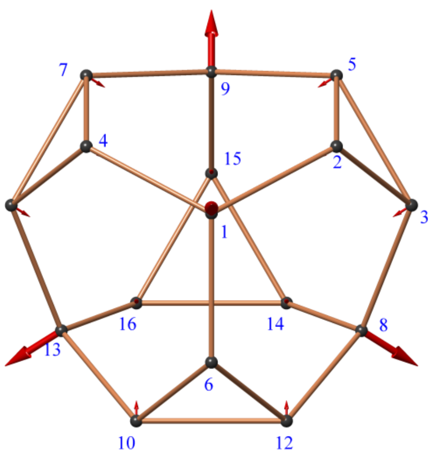


Figure 3. *A simple 'two-orbit' switch-pitch structure with tetrahedral symmetry (Group T_d)*

Nodes 1, 8, 9, and 13 are in one orbit of nodes, and the other nodes are in another orbit. The structure originates from the chamfered tetrahedron, and belongs to the symmetry group T_d.

We evaluate equation (2) for the T_d switch-pitch structure, where the characters of $\Gamma(v)$, $\Gamma_{\parallel}(e)$, Γ_f, and the results of $\Gamma(m)-\Gamma(s)$ under each symmetry operation are listed in Table 1.

Table 1. *Evaluation of relative mobility for the T_d switch-pitch structure*

T_d	E	$8C_3$	$3C_2$	$6S_4$	$6\sigma_d$
$\Gamma(v)$	16	1	0	0	4
$\Gamma_{\parallel}(e)$	24	0	0	0	2
Γ_f	48	0	0	0	4
$\Gamma(m)-\Gamma(s)$	−6	0	2	0	4

Through equation (3) and the final row of Table 1, $\Gamma(m)-\Gamma(s)$ can be expressed as:

$$\Gamma(m)-\Gamma(s) = A_1 - A_2 - 2T_1 \quad (8)$$

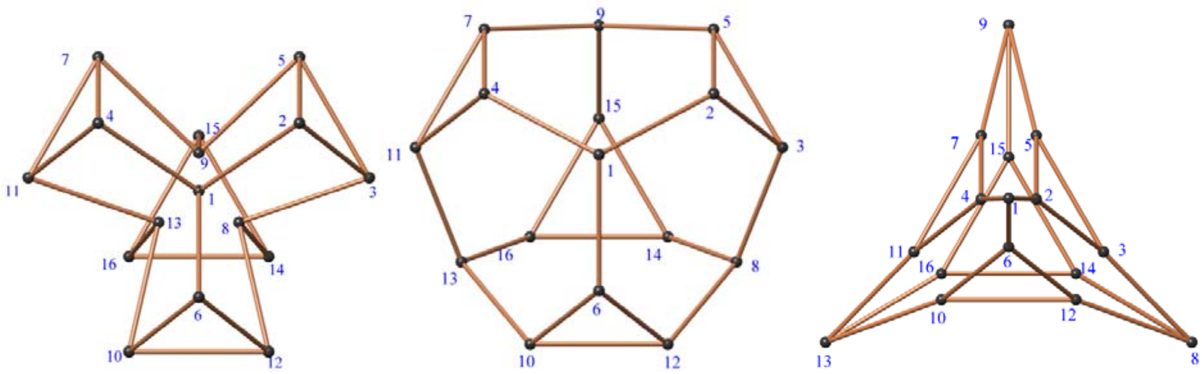
As the rank of the compatibility matrix is $r = 89$, and $m = 6\times16 - r - 6 = 1$, only one mechanism exists. We can separate $\Gamma(m)$ and $\Gamma(s)$ from the right side of equation (8), that is

$$\Gamma(m) = A_1, \text{ and } \Gamma(s) = A_2 + 2T_1 \quad (9)$$

This implies that the structure must have a finite mechanism along the path of A_1 symmetry (see the local motions expressed by arrows in Figure 3), because no equisymmetric state of self-stress exists [6]. Figure 4 shows some of the transformable configurations of the structure. We can notice that the structure does indeed retain the full symmetry.

4.2 Switch-Pitch Structure with Point Group Symmetry O_h

The switch-pitch structure shown in Figure 5 consists of 32 nodes and 48 members, and it has the symmetry properties of the group O_h. The nodes of Type A (or Type B) are in the same orbit of nodes, and they can be moved to be coincident with each other by the symmetry operations.



(a) *(b)* *(c)*

Figure 4. *Different configurations (a-c) along the folding path*

Table 2. *Evaluation of relative mobility for the O_h switch-pitch structure*

O_h	E	$8C_3$	$6C_2$	$6C_4$	$3C_2$	i	$6S_4$	$8S_6$	$3\sigma_h$	$6\sigma_d$
$\Gamma(v)$	32	2	0	0	0	0	0	0	0	8
$\Gamma_{\parallel}(e)$	48	0	0	0	0	0	0	0	−8	4
Γ_f	96	0	0	0	0	0	0	0	0	8
$\Gamma(m)-\Gamma(s)$	−6	0	2	−2	2	0	0	0	0	8

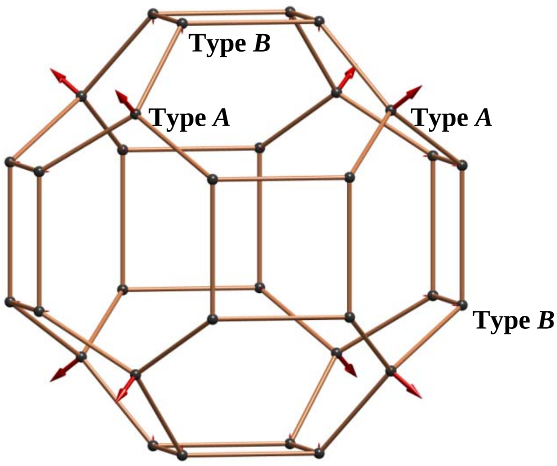


Figure 5. *A 'two-orbit' switch-pitch structure with cubic symmetry (Group O_h)*

Calculating the relative mobility of the O_h switch-pitch structure (refer to Table 2), we can write the results of $\Gamma(m)-\Gamma(s)$ as

$$\Gamma(m)-\Gamma(s)=A_{1g}+T_{2g}+A_{2u}-A_{2g}-2T_{1g}-A_{1u}-T_{2u} \tag{10}$$

The rank of the compatibility matrix for this structure is $r=181$, and $m=6\times 32-r-6=5$, revealing that

$$\Gamma(m)=A_{1g}+T_{2g}+A_{2u}, \text{ and}$$
$$\Gamma(s)=A_{2g}+2T_{1g}+A_{1u}+T_{2u} \tag{11}$$

which shows that the structure has a mechanism of A_{1g} with full symmetry (see the local motions for nodes plotted in Figure 5). Using the predictor-corrector algorithm, we verify that the mechanism along the path of A_{1g} symmetry is finite. Different geometric configurations along the folding path of A_{1g} symmetry are shown in Figure 6.

5. CONCLUSION

We have developed a generalized class of transformable switch-pitch structures, which are formed by two orbits of nodes and have high symmetry. Structures belonging to two different symmetry groups (i.e., T_d, and O_h) have been presented and verified to be mobile. The continuous and dramatic transformations of these structures may provide useful applications for deployable structures. Moreover, the technique described in this paper can identify the mobility and predict the kinematic behavior of internal mechanisms for a general switch-pitch structure.

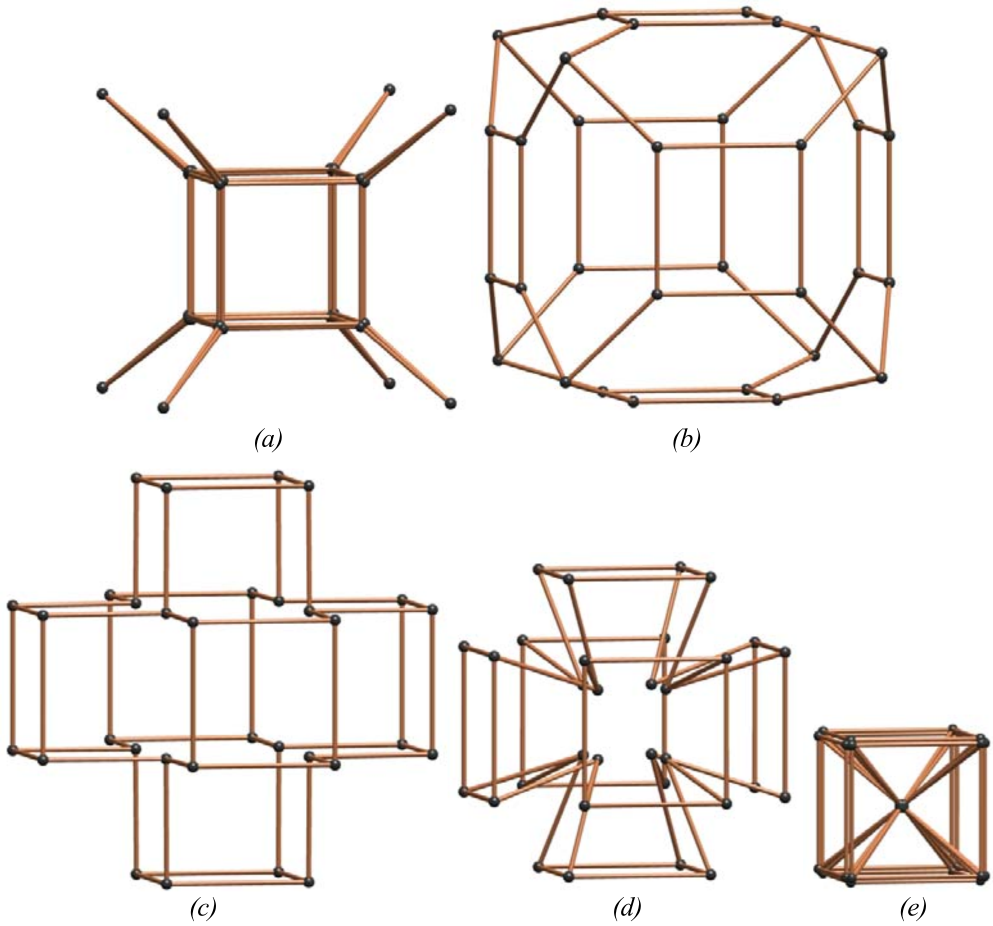


Figure 6. *An O_h switch-pitch structure, showing five typical configurations (from a to e) along the folding path*

6. ACKNOWLEDGEMENTS

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