Chapter 11

Generating stable symmetric tensegrities

11.1 Introduction

In this chapter we introduce and explain our catalogue, which is a computer program that can show you over a hundred different tensegrities that are "highly symmetric" and which the user can choose parameters to change its shape. Here we explain some of the group theory that one needs to understand how to use the program as well as to understand the symmetry of the tensegrity. Some examples of the catalogue can be seen in Subsection 11.7.4. One can access the program at:

`http://symmetric-tensegrity.com`

11.2 Symmetric tensegrities.

We consider a configuration $\mathbf{p} = [\mathbf{p}_1; \dots; \mathbf{p}_n]$ in d-dimensional space, where there is a group $\mathcal{G}$ of symmetries on the set of points of $\mathbf{p}$. We only insist, though, that $\mathcal{G}$ be a subgroup of all of the symmetries of the configuration. One way of saying this is that $\mathcal{G}$ *acts* on the configuration $\mathbf{p}$. For example, in Figure 11.1 the dihedral group $\mathcal{D}_3$ acts on the six points indicated. But we could just as well consider the cyclic group $\mathcal{C}_3$ as acting on the same six points.

If the configuration is part of a tensegrity, then we also insist that the cables are transformed to cables, struts to struts, and bars to bars under the group operations.

The notation for this situation is that if $\mathbf{p}_i$ is a point in the configuration $\mathbf{p}$ and g is a group element in $\mathcal{G}$, then $g\mathbf{p}_i$ is the image of the point $\mathbf{p}_i$ under the action of the group element g. If $\{g_i, g_j\}$ represents a cable (or a strut or bar), then under this convention, the action of the group element g on the cable is the unordered set $\{gg_i, gg_j\}$ representing the cable (strut or bar).

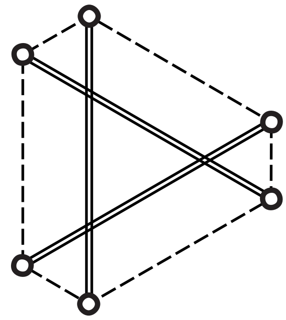


Figure 11.1: A symmetric tensegrity in the plane

11.3 Some group definitions

When a group $\mathcal{G}$ acts on a configuration of points $\mathbf{p}$, or any set for that matter, it is helpful to make a few definitions that help us understand the situation. We say that $\mathcal{G}$ acts *freely* on $\mathbf{p}$ if for all $\mathbf{p}_i$ in $\mathbf{p}$, $g\mathbf{p}_i = \mathbf{p}_i$ implies that the group element g is the identity.

In the example with Figure 11.1, the dihedral group $\mathcal{D}_3$ acts freely on the configuration of 6 points — as does the cyclic group $\mathcal{C}_3$.

We say that a subset T of the configuration $\mathbf{p}$ is a *transitivity class* if $T = \{g\mathbf{p}_i | g \text{ in } \mathcal{G}\}$ for some $\mathbf{p}_i$ of the configuration. A transitivity class is sometimes called an *orbit*. Note that transitivity classes form a partition of the points of the configuration — two transitivity classes are either disjoint, or the same. Similarly we define transitivity classes of cables and struts. If there is only one transitivity class (of vertices, say) then we say that $\mathcal{G}$ is *transitive* (on the vertices) or *acts transitively*.

In the example of Figure 11.1, $\mathcal{D}_3$ acts transitively, and there are two transitivity classes of cables, and one transitivity class of struts.

11.3.1 Highly symmetric tensegrities

One interesting special class of tensegrities are those where a group $\mathcal{G}$ of symmetries acts transitively and freely. This means that there is a one-to-one correspondence between the elements of $\mathcal{G}$ and the vertices $\mathbf{p} = [\mathbf{p}_1; \dots; \mathbf{p}_n]$ of the tensegrity. Indeed choose any vertex $\mathbf{p}_1$ and identify it with the identity element 1 of $\mathcal{G}$. If we enumerate the elements of $\mathcal{G} = (g_1 = 1, g_2, \dots, g_n)$, then g_i is identified with $g_i\mathbf{p}_1 = \mathbf{p}_i$, the i-th vertex of the configuration $\mathbf{p}$. For these tensegrities, we will show here that we can write the stress matrix in terms of the right regular representation permutation matrix.

First we need to find a symmetric self-stress. Suppose that $\boldsymbol{\omega} = [\dots; \omega_{ij}; \dots]$ is a proper self-stress for the tensegrity $(G, \mathbf{p})$, and $\mathcal{G}$ is a finite group acting freely and transitively on the vertices of $\mathbf{p}$, as described above. Further, we will assume that the associated stress matrix $\boldsymbol{\Omega}$ is positive semi-definite of rank $n - d - 1$. Now, without changing the rank or the positive semi-definiteness of the stress matrix, we can replace $\boldsymbol{\omega}$ and thus $\boldsymbol{\Omega}$ with the average stress $\Sigma_{g\in\mathcal{G}} g\boldsymbol{\omega}$, where $g\omega_{ij} = \omega_{kl}$ when the image of the member $\{ij\}$ is the member $\{kl\}$ under the action of g, and $g\boldsymbol{\omega} = [\dots; g\omega_{ij}; \dots]$. Since the sum of positive semi-definite quadratic forms is positive semi-definite, we have not introduced any negative eigenvalues in the stress matrix, and since the rank cannot decrease in the averaging process, we have not changed the rank either. But the action of $\mathcal{G}$ is now invariant on the force coefficients. In other words, $g\omega_{ij} = \omega_{ij}$, for all members $\{ij\}$ of G. So from now on we will assume that $\mathcal{G}$ is invariant on the force coefficients $\boldsymbol{\omega}$ and, of course, on the stress matrix $\boldsymbol{\Omega}$.

Consider one transitivity class of, say, cables. It has a representative of the form $\{1, c\}$ for c in $\mathcal{G}$ and we can identify the group element c with that transitivity class of cables. So every cable in that transitivity class appears in the group $\{g, gc\}$ as g varies over the elements of $\mathcal{G}$. Note the class will include the cable $\{c^{-1}, c^{-1}c\} \equiv \{1, c^{-1}\}$, so we an also identify the group element c^{-1} with the same transitivity class of cables.

Let $\rho_R(c)$ denote the permutation matrix that corresponds to right multiplication by c^{-1} in the right regular representation. This implies that $\rho_R(c)e_i = e_j$ if and only if $g_i c^{-1} = g_j$, for the standard basis vectors e_i, $i = 1, \ldots, n$ of Euclidean n-space. In other words, the (j, i) entry of $\rho_R(c)$, which is $e_j^{\mathrm{T}} \rho_R(c) e_i$, is 1 if and only if $g_i c^{-1} = g_j$, otherwise it is 0.

Note that $\rho_R(c)$ is not necessarily a symmetric matrix. However, if $g_i c^{-1} = g_j$ and $g_j c^{-1} = g_i$, then substituting we get $g_i c^{-1} c^{-1} = g_i$, which implies that $c^2 = 1$ in $\mathcal{G}$. And if $c^2 = 1$ in $\mathcal{G}$, $\rho_R(c)$ is symmetric, otherwise it is not. Furthermore, no diagonal entry of $\rho_R(c)$ is 1 unless $c = 1$.

Consider two possibilities. Suppose $c^2 = 1$ in $\mathcal{G}$, and there is a cable from p_1 to p_2, where $p_2 = cp_1$, c in $\mathcal{G}$. Then there is a cable from p_i to p_j if and only if $g_i c^{-1} = g_j$ (or equivalently $g_i c = g_j$), and there is a 1 in the (i, j) and (j, i) entries of $\rho_R(c)$, and all the other entries are 0.

Alternatively, suppose $c^2 \neq 1$ in $\mathcal{G}$, and there is a cable from p_1 to p_2, where $p_2 = cp_1$, c in $\mathcal{G}$. Then there is a cable from from p_i to p_j if and only if $g_i c = g_j$ (or equivalently $g_i = g_j c^{-1}$) if and only if there is a 1 in the (i, j) entry of $\rho_R(c)$. So the matrix $\rho_R(c) + \rho_R(c^{-1}) = \rho_R(c) + \rho_R(c)^{\mathrm{T}}$ has a 1 in the (i, j) entry and the (j, i) entry if and only if there is a cable between p_i and p_j.

Now consider the matrix

$$\boldsymbol{\Omega}(c) = \begin{cases} I - \rho_R(c) & \text{if } c^2 = 1; \\ 2I - (\rho_R(c) + \rho_R(c^{-1})) = 2I - (\rho_R(c) + \rho_R(c)^{\mathrm{T}}) & \text{otherwise} \end{cases} \tag{11.3.1}$$

where I is the n-by-n identity matrix. For this matrix it is clear that it is symmetric, the row and column sums are 0 (since this is true for $I - \rho_R(c)$, $\rho_R(c)$ being a permutation matrix), and the (i, j) and (j, i) entries are -1 if and only if there is cable between p_i and p_j. So $\boldsymbol{\Omega}(c)$ is a stress matrix with a force coefficient of 1 on the cables associated to the group element c.

Note that the same definition and properties hold if c is a strut instead of a cable.

We now describe the stress matrix for a tensegrity that has a group $\mathcal{G}$ operate freely and transitively on it. Choose one vertex, say p_1, in the configuration and identify that vertex with the identity 1 in $\mathcal{G}$. Then consider $c_1, c_2, \ldots, c_a$ in $\mathcal{G}$ that correspond to the transitivity classes of cables in the tensegrity, and $s_1, s_2, \ldots, s_b$ that correspond to the transitivity classes of struts in the tensegrity. So, for example, there is a cable corresponding to the transitivity class c_k between p_i and p_j if and only if $g_i c_k = g_j$ or $g_i c_k^{-1} = g_j$.

Define $\omega_k = \omega_{1i} > 0$, for $k = 1, \ldots, a$, where $\{p_1, p_i\}$ corresponds to the transitivity class of cables given by c_k, and ω is the starting equilibrium stress for the configuration $\mathbf{p}$. Similarly, define $\omega_{-k} < 0$ for $k = 1, \ldots, b$ for the struts.

The result of the definitions and the discussion above is that

$$\mathbf{\Omega} = \sum_{k=1}^{a} \omega_k \mathbf{\Omega}(c_k) + \sum_{k=1}^{b} \omega_{-k} \mathbf{\Omega}(s_k). \tag{11.3.2}$$

11.4 Irreducible components.

Once the right regular representation of the group $\mathcal{G}$ is obtained, each of the terms in (11.3.2) is an element of the right regular representation. Let $\mathbf{P}_R$ be the matrix that conjugates the right regular representation to the direct sum of irreducible representations. In other words $\mathbf{P}_R \rho_R \mathbf{P}_R^{-1}$ is the direct sum of irreducible representations as described in Theorem 9.5.4. By restricting to the subspace corresponding to each summand of the decomposition of Theorem 9.5.4, we can replace each term ρ_R in the definition of $\mathbf{\Omega}(c)$ and in Equation 11.3.2 with any irreducible representation. If ρ_i, $i = 1, \ldots, m$ are the irreducible representations of the group $\mathcal{G}$, then we define

$$\mathbf{\Omega}_i(c) = \begin{cases} I - \rho_i(c) & \text{if } c^2 = 1; \\ 2I - (\rho_i(c) + \rho_i(c^{-1})) = 2I - (\rho_i(c) + \rho_i(c)^{\mathrm{T}}) & \text{otherwise} \end{cases} \tag{11.4.1}$$

and similar to Equation 11.3.2 we define the *local stress matrix* for representation i as

$$\mathbf{\Omega}_i = \sum_{k=1}^{a} \omega_k \mathbf{\Omega}_i(c_k) + \sum_{k=1}^{b} \omega_{-k} \mathbf{\Omega}_i(s_k). \tag{11.4.2}$$

We have effectively block-diagonalised the stress matrix, with d blocks for each d-dimensional irreducible representation, which could thus be written as

$$\mathbf{P}_R \mathbf{\Omega} \mathbf{P}_R^{-1} = \mathbf{\Omega}_1 \oplus \sum_{j=1}^{\dim(\rho_2)} \mathbf{\Omega}_2 \oplus \ldots \oplus \sum_{j=1}^{\dim(\rho_m)} \mathbf{\Omega}_m$$

This brings us to one of the main points of using representation theory. Instead of computing whether $\mathbf{\Omega}$ is positive semi-definite of the appropriate rank directly, we see that it is enough to compute whether each of the components $\mathbf{\Omega}_i$ is positive definite or positive semi-definite. Furthermore, it is possible to keep track of the rank of $\mathbf{\Omega}$ by keeping track of the rank of each $\mathbf{\Omega}_i$. This holds the possibility of greatly reducing the amount of computation, and allows the possibility of first deciding on the stress with some desired properties, and then calculating the configuration by determining the kernel of $\mathbf{\Omega}$, and even more easily by calculating the kernel of each $\mathbf{\Omega}_i$.

Let us label the trivial representation, which takes all the group elements into the identity, as the first representation ρ_1. Then we see that $\mathbf{\Omega}_1 = 0$. Suppose that only one other irreducible representation, say ρ_2, is such that $\mathbf{\Omega}_2$ is singular with a one dimensional kernel, and that all the other representations $\mathbf{\Omega}_i$ for $i = 3, \ldots, m$ are positive semi-definite. If the dimension of ρ_2 is d, then by Theorem 9.5.4, the ρ_2 representation appears exactly d times in the right regular representation ρ_R. So the kernel of $\mathbf{\Omega}$ is $d+1$ dimensional, and it is positive semi-definite. Under these conditions, it means that the associated tensegrity framework is super stable in all dimensions up to affine motions.

11.4.1 Example of the method.

Let us take $\mathcal{G}$ to be the dihedral group $\mathcal{D}_3$ with 6 elements, and the group elements corresponding to cables to be $c_1 = C_3\sigma$, $c_2 = C_3^2 S$ and $s_1 = \sigma$, using the notation of Subsection 9.5.9.

$$\rho_3(C_3) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}, \quad \rho_3(s_1) = \rho_2(\sigma) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We then calculate

$$\rho_3(c_1) = \rho_3(C_3\sigma) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \quad \rho_3(c_2) = \rho_2(C_3^2\sigma) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Using Equation 11.4.1 we get

$$\mathbf{\Omega}_3(c_1) = \begin{bmatrix} 3/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}, \quad \mathbf{\Omega}_3(c_2) = \begin{bmatrix} 3/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \quad \mathbf{\Omega}_3(s_1) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then the local stress matrix for the third representation is

$$\mathbf{\Omega}_3 = \begin{bmatrix} \frac{3}{2}(\omega_1 + \omega_2) & \frac{\sqrt{3}}{2}(-\omega_1 + \omega_2) \\ \frac{\sqrt{3}}{2}(-\omega_1 + \omega_2) & \frac{1}{2}(\omega_1 + \omega_2) + 2\omega_{-1} \end{bmatrix}.$$

The second representation is one-dimensional, so we can use the character χ_2 as in the character table in Subsection 9.5.9 to calculate the following.

$$\rho_2(c_1) = \begin{bmatrix} -1 \end{bmatrix}, \quad \rho_2(c_2) = \begin{bmatrix} -1 \end{bmatrix}, \quad \rho_2(s_1) = \begin{bmatrix} -1 \end{bmatrix}.$$

Using Equation 11.4.1 we get

$$\mathbf{\Omega}_2(c_1) = \begin{bmatrix} 2 \end{bmatrix}, \quad \mathbf{\Omega}_2(c_2) = \begin{bmatrix} 2 \end{bmatrix}, \quad \mathbf{\Omega}_2(s_1) = \begin{bmatrix} 2 \end{bmatrix}$$

Then the local stress matrix for the second representation is

$$\mathbf{\Omega}_2 = \begin{bmatrix} 2\omega_1 + 2\omega_2 + 2\omega_{-1} \end{bmatrix}.$$

We calculate the determinants of the local stress matrices, which we call the *i-th stress determinant* $\Delta_i = \Delta_i(\omega_1, \dots, \omega_a, \omega_{-1}, \dots, \omega_{-b})$, for representation i.

$$\Delta_3 = \det(\mathbf{\Omega}_3) = 3\omega_1\omega_2 + 3(\omega_1 + \omega_2)\omega_{-1}, \quad \Delta_2 = \det(\mathbf{\Omega}_3) = 2\omega_1 + 2\omega_2 + 2\omega_{-1}.$$

We can normalize the stresses so that $\omega_1 + \omega_2 = 1$. Then $\omega_2 = 1 - \omega_1$, and for $0 < \omega_1 < 1$ and $0 < 1 - \omega_1 = \omega_2 < 1$. Note that when ω_1, ω_2, and ω_{-1} are all positive $\mathbf{\Omega}_2$ and $\mathbf{\Omega}_3$ are both positive definite. Then allow to ω_{-1} to decrease and become negative while fixing ω_1 and $\omega_2 = 1 - \omega_1$. The polynomial Δ_3 first becomes 0, changing from positive to negative, when $\omega_{-1} = -\omega_1(1 - \omega_1) > -1$. The polynomial Δ_2 first becomes 0, changing from positive to negative, when $\omega_{-1} = -1$ — see Figure 11.2. Thus when $\omega_{-1} = -\omega_1(1 - \omega_1)$, $\mathbf{\Omega}_2$ is positive definite, $\mathbf{\Omega}_3$ is positive semi-definite with a 1-dimensional kernel, and $\mathbf{\Omega}_1 = 0$ but is 1-dimensional. Thus in the full right regular representation $\mathbf{\Omega}$ is positive semi-definite with a 3-dimensional kernel, since the 2-dimensional representation appears twice in the direct sum.

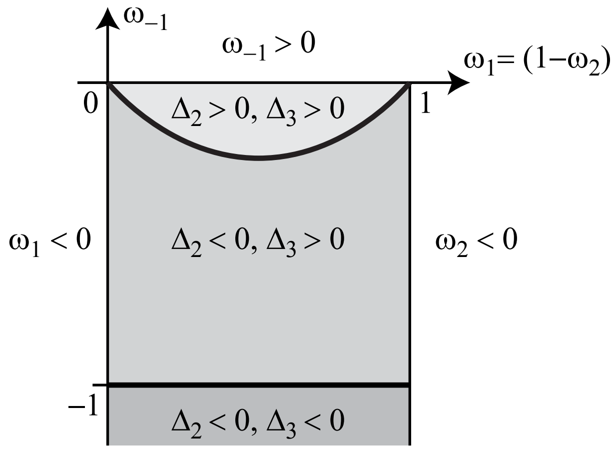


Figure 11.2: A determinant plot for the tensegrity shown in Figure 11.1. The vertical axis is the strut force coefficient ω_{-1}, and the horizontal axis shows the cable force coefficients ω_1 and ω_2, normalised so that $\omega_1 + \omega_2 = 1$. The shaded region shows where the force coefficients are proper, i.e. $\omega_{-1} < 0$, $\omega_1 > 0$, $\omega_2 > 0$. The shaded region is split into different regions depending on the signs of the stress determinants Δ_2 and Δ_3. The quadratic line shows where $\Delta_2 = 0$, and the straight line shows where $\Delta_3 = 0$ ($\Delta_1 = 0$ everywhere, by definition). We are interested in points where some stress determinants are zero but the rest are all positive: for any value of ω_1 this will be found by descending from the horizontal axis until the first $\Delta_i = 0$ line is crossed, defining the value of ω_{-1} for equilibrium to be satisfied.

Thus the associated tensegrity is (universally) rigid and prestress stable in all dimensions, since there are at least 3 distinct stressed directions.

Notice that we started with the graph of the framework only, not the actual configuration itself. Then we found the force coefficients corresponding to an equilibrium self-stress for some configuration. To find the configuration, we first find a vector in the kernel of the third local stress matrix $\mathbf{\Omega}_3$,

$$\mathbf{\Omega}_3 = \begin{bmatrix} \frac{3}{2} & \frac{\sqrt{3}}{2}(1-2\omega_1) \\ \frac{\sqrt{3}}{2}(1-2\omega_1) & \frac{1}{2}-2\omega_1(1-\omega_1) \end{bmatrix}.$$

For example, the following vector is in the kernel,

$$\begin{bmatrix} \frac{\sqrt{3}}{2}(1-2\omega_1) \\ -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = p_1.$$

From the definition of Equation 11.3.1 it is clear that if we let $p_i = \rho_3(g_i)p_1$, then the configuration $\mathbf{p} = [\mathbf{p}_1; \ldots; \mathbf{p}_n]$ will have an equilibrium self-stress with a 3-dimensional kernel, and be positive semi-definite as desired when $0 < \omega_1 < 1$. This gives a configuration as shown in Figure 11.1.

11.5 Groups for 3-dimensional examples.

As with any representation of a group, one must decide on the initial description of the group that will be represented into the group of matrices. For tensegrities in three-space, a natural choice is to use certain permutation groups as the initial groups. One reason for this is that permutations are unbiased as far as pointing to any particular representation. This is helpful since the process that we describe here will be such that different representations will be chosen for the configuration that displays the stress which has a stress matrix of maximal rank as well as being positive semi-definite. But an even more relevant reason for the permutation description is that the group multiplication is particularly convenient and efficient to calculate. Also, properties of the underlying graph of the tensegrity can be read off easily from the permutation description of the cable and strut generators c_i and s_i.

A number of the groups that we will use are formed from the direct product of a permutation group with the permutation group $\mathcal{S}_2$. For two groups $\mathcal{G}_1$ and $\mathcal{G}_2$, their *direct product*, written as $\mathcal{G}_1 \times \mathcal{G}_2$, is the set of pairs (g_1, g_2), and the group multiplication is given by $(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2')$, where g_1, g_1' are in $\mathcal{G}_1$, and g_2, g_2' are in $\mathcal{G}_2$. It is easy to check that the required properties of a group are satisfied by $\mathcal{G}_1 \times \mathcal{G}_2$. It might be useful to note that the permutation group $\mathcal{S}_2$ is isomorphic with $\mathbb{Z}_2$, which is the group $\{1, -1\}$ with group multiplication being the multiplication of real numbers.

We use the following six groups plus the dihedral groups, which will be described in later subsections. See Table 9.5 for further notes about these groups.

i.) The alternating group on 4 symbols $\mathcal{A}_4$. This is the group of even permutations of the symbols $\{1, 2, 3, 4\}$. A permutation is *even* if it can be written as an even number of transpositions, where a *transposition* interchanges exactly 2 symbols, leaving all the others fixed. If a permutation is not even, then it is call an *odd permutation*. It is a nice exercise to show that the even permutations form a group. The order of $\mathcal{A}_4$ is 12.

ii.) The symmetric group on 4 symbols $\mathcal{S}_4$. This is the group of all permutations of the symbols $\{1, 2, 3, 4\}$, and has order 24.

iii.) The alternating group on 5 symbols $\mathcal{A}_5$. This is the group of all even permutations of the symbols $\{1, 2, 3, 4, 5\}$ and has order 60.

iv.) The group $\mathcal{A}_4 \times \mathcal{S}_2$. It has order 24. Note that this group has the same order as $\mathcal{S}_4$, but it is not isomorphic to $\mathcal{S}_4$, since, for example, $\mathcal{S}_4$ has an element of order 3, whereas $\mathcal{A}_4 \times \mathcal{S}_2$ does not.

v.) The group $\mathcal{S}_4 \times \mathcal{S}_2$. It has order 48.

vi.) The group $\mathcal{A}_5 \times \mathcal{S}_2$. It has order 120.

11.6 Presentation of groups.

Another method that can be used to define a group is what is called a *presentation* of a group $\mathcal{G}$. This is a list of generators $a, b, \dots$ for $\mathcal{G}$, together with what are called *relations* $r_1, r_2, \dots$. These are finite products of the generators and their inverses, called *words*, such that each word is equal to the identity in $\mathcal{G}$. The group $\mathcal{G}$ is defined by this presentation in the sense that if any other group $\mathcal{H}$ has the property that it is has corresponding generators satisfying the same relations, then $\mathcal{H}$ is homomorphic to $\mathcal{G}$. The presentation is written as $\mathcal{G} = \{a, b, \cdots \mid r_1 = r_2 = \cdots = 1\}$.

As an example, the cyclic group $\mathcal{C}_n$ can be defined by the presentation with one generator a, and one relation $a^n = 1$, so $\mathcal{C}_n = \{a \mid a^n = 1\}$. For another example the alternating group $\mathcal{A}_5$ has the presentation, $\mathcal{A}_5 = \{a, b \mid a^2 = b^3 = (ab)^5 = 1\}$.

11.7 Representations for groups of interest

To generate tensegrities, we need all the irreducible representation of the group we are working with, even though we may be interested only in the 3-dimensional (or possibly the 2-dimensional) representations. We give some of the ideas here that we use to generate those irreducible representations.

Here we will give concrete examples of matrices $\rho_i(g)$ that form an irreducible representation for the dihedral groups, and for each of the groups listed in Section 11.5. In order to compute any element $\rho_i(g)$ for g in $\mathcal{G}$ it is enough to do it for the generators $g_1, \dots$ .We have included the character tables for all of the groups that we are considering, and this can be used to check the correctness of the choices that we will describe next for the g_i.

11.7.1 Dihedral groups.

The dihedral group $\mathcal{D}_n$ can be defined as the full group of rotational symmetries of the regular polygon in space. It is not too hard to see that $\mathcal{D}_n$ has the presentation $\mathcal{D}_n = \{r, s \mid r^n = s^2 = (sr)^2 = 1\}$. Rotation by $2\pi/n$, denoted as C_n, corresponds to r, and rotation by π about some suitable axis in the plane of the polygon C_2 corresponds to s.

The following is the character table for $\mathcal{D}_n$. It is usual to distinguish the case when n is odd, and when n is even. In both cases the elements r and s generate $\mathcal{D}_n$.

Groups $\mathcal{D}_n$, n odd $n \geq 3$.

$\mathcal{D}_n$	1	$r^k (k = 1, \dots, \frac{n-1}{2})$	s
$\mathcal{D}_n = (\mathrm{n2})$	E	C_n^k	C_2
$A_1 = \chi_1$	1	1	1
$A_2 = \chi_2$	1	1	-1
$E_j = \psi_j$	2	$2\cos(\frac{2\pi jk}{n})$	0
$(j = 1, \dots, \frac{n-1}{2})$			

There are $(n+1)/2$ conjugacy classes and irreducible representations. It is clear that r and s generate $\mathcal{D}_n$, and that $r^a s^b$; $a = 0, 1, \dots, n-1$; $b = 0, 1$ enumerate $\mathcal{D}_n$.

Since the characters χ_1 and χ_2 are one-dimensional, we can regard them as representations. Let $\rho_j, j = 1, \dots, \frac{n-1}{2}$ denote the representation corresponding to the character ψ_j. Then we define

$$\rho_j(r) = \begin{bmatrix} \cos(\frac{2\pi j}{n}) & -\sin(\frac{2\pi j}{n}) \\ \sin(\frac{2\pi j}{n}) & \cos(\frac{2\pi j}{n}) \end{bmatrix}, \ \rho_j(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to check that the trace of each of these matrices is the character of the corresponding group element.

Groups $\mathcal{D}_n$, n even $n \geq 4$.

$\mathcal{D}_n$	1	$r^k (k = 1, \dots, \frac{n}{2} - 1)$	$r^{n/2}$	s	rs
$\mathcal{D}_n = (\mathrm{n22})$	E	C_n^k	$C_n^{n/2}$	C_2'	C_2''
$A_1 = \chi_1$	1	1	1	1	1
$A_2 = \chi_2$	1	1	1	-1	-1
$B_1 = \chi_3$	1	$(-1)^k$	$(-1)^{n/2}$	1	-1
$B_2 = \chi_4$	1	$(-1)^k$	$(-1)^{n/2}$	-1	1
$E_j = \psi_j$	2	$2\cos(\frac{2\pi jk}{n})$	$2(-1)^{n/2}$	0	0
$(j = 1, \dots, \frac{n}{2} - 1)$					

There are $n/2 + 3$ conjugacy classes and irreducible representations. Again r and s generate $\mathcal{D}_n$, and $r^a s^b$; $a = 0, 1, \dots, n-1$; $b = 0, 1$ enumerate $\mathcal{D}_n$.

Since the characters $\chi_1, \chi_2, \chi_3, \chi_4$ are one-dimensional, we can regard them as representations. Let $\rho_j, j = 1, \dots, \frac{n}{2} - 1$ denote the representation corresponding to the character ψ_j. Then, as before, we define

$$\rho_j(r) = \begin{bmatrix} \cos(\frac{2\pi j}{n}) & -\sin(\frac{2\pi j}{n}) \\ \sin(\frac{2\pi j}{n}) & \cos(\frac{2\pi j}{n}) \end{bmatrix}, \ \rho_j(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is again easy to check that the trace of each of these matrices is the character of the corresponding group element.

11.7.2 Basic non-dihedral representations.

In each of the cases below we have indicated that products of the generators $g_1^a g_2^b g_3^c \dots$ can be used to enumerate the group $\mathcal{G}$. For example for the group $\mathcal{S}_4$, each g_i permutes the set $\{1, \dots, i\}$. So the same is true for $g_1^a g_2^b \dots g_i^c$. So if we are given any g in $\mathcal{G}$ as a permutation, choose the integer c so that g_i^{-c} permutes the symbol i to the same symbol as g. Then c is the exponent of g_i in the decomposition $g = g_1^a g_2^b \ldots g_3^c$. Continuing this way one can easily determine all the exponents $a, b, \ldots c$. A similar idea works for the other groups $\mathcal{A}_4$ and $\mathcal{A}_5$ as well.

Group $\mathcal{S}_4$.

$\mathcal{S}_4$	1	(123)	$(12)(34)$	(1234)	(12)
$O=(432)$	E	$8C_3$	$3C_2$	$6C_4$	$6C_4'$
$T_d=(43m)$	E	$8C_3$	$3C_2$	$6S_4$	$6\sigma_d$
$A_1=\chi_1$	1	1	1	1	1
$A_2=\chi_2$	1	1	1	-1	-1
$E=\chi_3$	2	-1	2	0	0
$T_1=\chi_4$	3	0	-1	1	-1
$T_2=\chi_5$	3	0	-1	-1	1

We describe all the elements of $\mathcal{S}_4$ so that they can be enumerated easily with the following generators. Let

$$g_1 = (12),\ g_2 = (123),\ g_3 = (1234)$$

in disjoint cycle notation. Then it is easy to check that $g_1^a g_2^b g_3^c$, for $a = 0, 1,\ b = 0, 1, 2,\ c = 0, 1, 2, 3$, are all distinct. Thus these are the $2 \cdot 3 \cdot 4 = 24$ elements of $\mathcal{S}_4$.

There are two irreducible 3-dimensional representations of $\mathcal{S}_4$. They both can be thought of as symmetries of the unit 3-dimensional cube. One representation ρ_4 is the set of rotations that are symmetries of the cube. Use the numbers $1, 2, 3, 4$ to label the vertices of the cube, where opposite vertices have the same label.

$$1 \to \pm \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\quad 2 \to \pm \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\quad 3 \to \pm \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\quad 4 \to \pm \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

It is easy to check that any rotation of the cube corresponds to a permutation of these labels, and that this correspondence is a group isomorphism from $\mathcal{S}_4$ to the rotations of the cube. We calculate the matrices implied by the above correspondence as follows.

$$\rho_4(g_1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\quad \rho_4(g_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix},\quad \rho_4(g_3) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

For the representation ρ_5 we make the correspondence between permutation elements and a subset of the vertices of the cube that form a regular tetrahedron. Then ρ_5 corresponds to the full symmetry group of the tetrahedron. To make definite the correspondence we define the following correspondence.

$$1 \to \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\quad 2 \to \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix},\quad 3 \to \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix},\quad 4 \to \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

We calculate the matrices implied by the above correspondence as follows.

$$\rho_5(g_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \rho_5(g_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \quad \rho_5(g_3) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

From these correspondences it is possible to look at any particular symmetry operation, and be able to determine the corresponding permutation in $\mathcal{S}_4$.

Now consider the 2-dimensional irreducible representation ρ_3 of $\mathcal{S}_4$. It is possible to check that the following correspondence will work.

$$\rho_3(g_1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho_3(g_2) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}, \quad \rho_3(g_3) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}.$$

(The group elements $g_1 = (12)$, $g_2 = (123)$ generate the symmetric group S_3, which has a 2-dimensional representation which are the transformation matrices corresponding to symmetry operations acting on a triangle. From the character table, we can determine that $\rho_3((12)(34)) = 1$ and so $\rho_3((12)) = \rho_3((34))$, and thus $\rho_3((1234)) = \rho_3((123)(34)) = \rho_3((123))\rho_3((34)) = \rho_3((123))\rho_3((12))$.)

The non-trivial 1-dimensional representation ρ_2 of $\mathcal{S}_4$, can be read from the character table, observing that it is $[1]$ for even permutations, and $[-1]$ for odd permutations.

$$\rho_2(g_1) = \begin{bmatrix} -1 \end{bmatrix}, \quad \rho_2(g_2) = \begin{bmatrix} 1 \end{bmatrix}, \quad \rho_2(g_3) = \begin{bmatrix} -1 \end{bmatrix}.$$

Group $\mathcal{A}_4$.

$\mathcal{A}_4$ / $T = (23)$	1 / E	(123) / $4C_3$	(132) / $4C_3^2$	$(13)(24)$ / $3C_2$
$A = \chi_1$	1	1	1	1
${}^1E = \chi_2$	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	1
${}^2E = \chi_3$	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	1
$T = \chi_4$	3	0	0	-1

Since the alternating group of even permutations on four symbols $\mathcal{A}_4$ is a subgroup of $\mathcal{S}_4$, most of the work for finding the irreducible representations is done. One new feature is that this group has two non-trivial irreducible one-dimensional complex representations. When

$$g_1 = (123), \; g_2 = (12)(34), \; g_3 = (13)(24),$$

then $g_1^a g_2^b g_3^c$ for $a = 0, 1, 2, \; b = 0, 1, \; c = 0, 1$ enumerate all the elements of $\mathcal{A}_4$, similar to the case for $\mathcal{S}_4$.

For the 3-dimensional irreducible representation ρ_4 of $\mathcal{A}_4$, we can simply restrict to the subset of either irreducible 3-dimensional representation of $\mathcal{S}_4$. So we get the following.

$$\rho_4(g_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \quad \rho_4(g_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \rho_4(g_3) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The other two irreducible representations ρ_2 and ρ_3, which are complex, are the following.

$$\rho_2(g_1) = \left[-1/2 + i\sqrt{3}/2\right], \ \ \rho_2(g_2) = \left[1\right], \ \ \rho_2(g_3) = \left[1\right],$$

and

$$\rho_3(g_1) = \left[-1/2 - i\sqrt{3}/2\right], \ \ \rho_3(g_2) = \left[1\right], \ \ \rho_3(g_3) = \left[1\right].$$

Note that when these complex representations appear in the stress matrix, the imaginary parts vanish because each time a group element appears, so does its inverse with the same real coefficient and the representation for the inverse is the complex conjugate.

Group $\mathcal{A}_5$.

$\mathcal{A}_5$ / I	1 / E	(12345) / $12C_5$	(13524) / $12C_5^2$	(123) / $20C_3$	$(12)(34)$ / $15C_2$
$A = \chi_1$	1	1	1	1	1
$T_1 = \chi_2$	3	τ	τ'	0	-1
$T_2 = \chi_3$	3	τ'	τ	0	-1
$G = \chi_4$	4	-1	-1	1	0
$H = \chi_5$	5	0	0	-1	1

We use the numbers $\tau = (1 + \sqrt{5})/2$ (the golden ratio) and $\tau' = (1 - \sqrt{5})/2$. Note that $\tau\tau' = -1$, $\tau + \tau' = 1$, $\tau^2 - \tau - 1 = (\tau')^2 - \tau' - 1 = 0$. It is also worth noting that $\cos(2\pi/5) = -\tau'/2$ and $\cos(4\pi/5) = -\tau/2$.

Similar to the previous cases, we define the following group elements.

$$g_1 = (123), \ g_2 = (12)(34), \ g_3 = (13)(24), \ g_4 = (12345)$$

Then $g_1^a g_2^b g_3^c g_4^d$, for $a = 0, 1, 2, \ b = 0, 1, \ c = 0, 1, \ d = 0, 1, 2, 3, 4$, enumerate $\mathcal{A}_5$.

To find the 3-dimensional representations of the permutation group $\mathcal{A}_5$, following the ideas used for $\mathcal{S}_4$, we partition the vertices of the regular dodecahedron into 5 sets of 4 vertices, where each of those sets form a regular tetrahedron, and the tetrahedra are permuted by any rotation of the dodecahedron.

The following are one of two choices for the vertices of the dodecahedron partitioned into the 5 tetrahedra.

$$\begin{aligned}
T_1 &= \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \tau \\ -\tau' \end{bmatrix}, \begin{bmatrix} -\tau' \\ 0 \\ -\tau \end{bmatrix}, \begin{bmatrix} -\tau \\ \tau' \\ 0 \end{bmatrix} \right\}, T_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\tau \\ -\tau' \end{bmatrix}, \begin{bmatrix} \tau' \\ 0 \\ -\tau \end{bmatrix}, \begin{bmatrix} \tau \\ -\tau' \\ 0 \end{bmatrix} \right\}, \\
T_3 &= \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\tau \\ \tau' \end{bmatrix}, \begin{bmatrix} -\tau' \\ 0 \\ \tau \end{bmatrix}, \begin{bmatrix} -\tau \\ -\tau' \\ 0 \end{bmatrix} \right\}, T_4 = \left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \tau \\ \tau' \end{bmatrix}, \begin{bmatrix} \tau' \\ 0 \\ \tau \end{bmatrix}, \begin{bmatrix} \tau \\ \tau' \\ 0 \end{bmatrix} \right\}, \\
T_5 &= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}.
\end{aligned}$$

We identify each tetrahedron T_i with i, for $i = 1, 2, 3, 4, 5$, and any rotation of the dodecahedron corresponds uniquely to an even permutation of the set $\{1, 2, 3, 4, 5\}$. Then it is easy to calculate that the corresponding representation on the generators g_j defined above.

$$\rho_3(g_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad \rho_3(g_2) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\rho_3(g_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad , \quad \rho_3(g_4) = \frac{1}{2}\begin{bmatrix} 1 & \tau' & \tau \\ -\tau' & -\tau & -1 \\ \tau & 1 & \tau' \end{bmatrix}.$$

We can check that this is the representation ρ_3 in the character table by computing traces of the appropriate matrices above. In particular $\chi_3(g_4) = \tau'$.

For the representation ρ_2, we observe that if we interchange τ and τ' everywhere (which corresponds to the other possible choice for partitioning the vertices between tetrahedra), then we get another representation as described by the following generators.

$$\rho_2(g_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad \rho_2(g_2) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\rho_2(g_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad , \quad \rho_2(g_4) = \frac{1}{2}\begin{bmatrix} 1 & \tau & \tau' \\ -\tau & -\tau' & -1 \\ \tau' & 1 & \tau \end{bmatrix}.$$

It is clear that this is the representation corresponding to χ_2, since $\chi_2(g_4) = \tau$

To find the four-dimensional representation ρ_4, we can consider the transformation matrices of a regular 4-dimensional simplex (consisting of 5 vertices) in 4-space corresponding to even permutations of the vertices. We choose the following vectors for the vertices of the 4 simplex (centred on the origin).

$$1 \to \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1/\sqrt{5} \end{bmatrix}, \quad 2 \to \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1/\sqrt{5} \end{bmatrix}, \quad 3 \to \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1/\sqrt{5} \end{bmatrix}, \quad 4 \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1/\sqrt{5} \end{bmatrix}, \quad 5 \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4/\sqrt{5} \end{bmatrix}.$$

We then calculate the following matrices for the generators above.

$$\rho_4(g_1) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \rho_4(g_2) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\rho_4(g_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \rho_4(g_4) = \frac{1}{4}\begin{bmatrix} -3 & 1 & 1 & -\sqrt{5} \\ 1 & 1 & -3 & -\sqrt{5} \\ -1 & 3 & -1 & \sqrt{5} \\ \sqrt{5} & \sqrt{5} & \sqrt{5} & -1 \end{bmatrix}.$$

Note that if we rescale the last coordinate of each of the vectors in the 4-simplex above by replacing $\sqrt{5}$ by 1, then the corresponding representation will have all rational entries, but will not be into the orthogonal group. This implies that some of the polynomials calculated later will eventually have only rational coefficients.

For the irreducible 5-dimensional representation ρ_5, we can calculate the following matrices for the g_i's.

$$\rho_5(g_1) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \rho_5(g_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\rho_5(g_3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho_5(g_4) = \begin{bmatrix} \frac{1-3\sqrt{5}}{16} & \frac{-1-\sqrt{5}}{16} & \frac{1+\sqrt{5}}{8} & -\frac{1}{4} & \frac{1-\sqrt{5}}{8} \\ \frac{-3-3\sqrt{5}}{16} & \frac{-1+3\sqrt{5}}{16} & \frac{-3+\sqrt{5}}{8} & -\frac{\sqrt{5}}{4} & \frac{3+\sqrt{5}}{8} \\ \frac{-3-3\sqrt{5}}{8} & \frac{3-\sqrt{5}}{8} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{4} & -\frac{\sqrt{5}}{4} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{-3+3\sqrt{5}}{8} & \frac{-3-\sqrt{5}}{8} & -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}.$$

One warning about this representation is that not all the matrices are orthogonal. (In particular $\rho_5(g_4)$ is not orthogonal.)

The representation ρ_5 was constructed by taking first what is called the tensor product $\rho_2 \otimes \rho_2$ and then calculating what is called the symmetric component. This turns out to be equivalent to the 6-dimensional representation that is the sum $\rho_5 + \rho_1$ of the 5-dimensional representation ρ_5 we are looking for and the trivial representation ρ_1. We can project this 6-dimensional representation onto ρ_5 easily.

To check that the matrices have been calculated correctly, one can observe that $\mathcal{A}_5 = \{a, b \mid a^2 = b^2 = (ab)^5 = 1\}$ is a presentation of $\mathcal{A}_5$. If one has a group $\mathcal{G}$, which is not the identity, and the relations in the presentation above are satisfied by some pair of the group elements a and b in $\mathcal{G}$, which generate all of $\mathcal{G}$, then $\mathcal{G}$ is isomorphic to the group $\mathcal{A}_5$. (It turns out that $\mathcal{A}_5$ has no non-trivial homomorphic group images, i.e. it is *simple*, so we know that any such group is isomorphic to $\mathcal{A}_5$.) In our case, we can take $a = (12)(34)$ and $b = (235)$ in disjoint cycle notation. Then $a = g_2$, $b = g_2 g_4$, $ab = g_4$. So we can verify that our matrices are chosen properly by checking the relations for the corresponding matrices. The characters are also easily checked.

Groups $\mathcal{A}_4 \times \mathcal{S}_2$, $\mathcal{S}_4 \times \mathcal{S}_2$, and $\mathcal{A}_5 \times \mathcal{S}_2$.

For the direct product with $\mathcal{S}_2$, there is the generator of order 2 that commutes with all of the elements of the group, which can be multiplied with all the other elements easily.

$\mathcal{A}_4 \times \mathcal{S}_2$	1	(123)	(132)	$(13)(24)$	-1	$-(132)$	$-(123)$	$-(13)(24)$
$T_h = (m3)$	E	$4C_3$	$4C_3^2$	$3C_2$	i	$4S_6$	$4S_6^2$	$3\sigma_d$
$A_g = \chi_1$	1	1	1	1	1	1	1	1
$E_g = \chi_2$	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	1	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	1
$E_g = \chi_3$	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	1	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	1
$T_g = \chi_4$	3	0	0	-1	3	0	0	-1
$A_u = \chi_5$	1	1	1	1	-1	-1	-1	-1
$E_u = \chi_6$	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	-1	-1	$-\frac{-1-i\sqrt{3}}{2}$	$-\frac{-1+i\sqrt{3}}{2}$	1
$E_u = \chi_7$	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	-1	-1	$-\frac{-1+i\sqrt{3}}{2}$	$-\frac{-1-i\sqrt{3}}{2}$	1
$T_u = \chi_8$	3	0	0	-1	-3	0	0	1

$\mathcal{S}_4 \times \mathcal{S}_2$	1	(123)	(12)	(1234)	$(12)(34)$	-1	$-(1234)$	$-(123)$	$-(12)(34)$	$-(12)$
$O_h = (\mathrm{m3m})$	E	$8C_3$	$6C_2$	$6C_4$	$3C_2$	i	$6S_4$	$8S_6$	$3\sigma_h$	$6\sigma_d$
$A_{1g} = \chi_1$	1	1	1	1	1	1	1	1	1	1
$A_{2g} = \chi_2$	1	1	-1	-1	1	1	-1	1	1	-1
$E_g = \chi_3$	2	-1	0	0	2	2	0	-1	2	0
$T_{1g} = \chi_4$	3	0	-1	1	-1	3	1	0	-1	-1
$T_{2g} = \chi_5$	3	0	1	-1	-1	3	-1	0	-1	1
$A_{1u} = \chi_6$	1	1	1	1	1	-1	-1	-1	-1	-1
$A_{2u} = \chi_7$	1	1	-1	-1	1	-1	1	-1	-1	1
$E_u = \chi_8$	2	-1	0	0	2	-2	0	1	-2	0
$T_{1u} = \chi_9$	3	0	-1	1	-1	-3	-1	0	1	1
$T_{2u} = \chi_{10}$	3	0	1	-1	-1	-3	1	0	1	-1

$\mathcal{A}_5 \times \mathcal{S}_2$	1	(12345)	(13524)	(123)	$(12)(34)$	-1	$-(13524)$	$-(12345)$	$-(123)$	$-(12)(34)$
I_h	E	$12C_5$	$12C_5^2$	$20C_3$	$15C_2$	i	$12S_{10}$	$12S_{10}^3$	$20S_6$	15σ
$A_g = \chi_1$	1	1	1	1	1	1	1	1	1	1
$T_{1g} = \chi_2$	3	τ	τ'	0	-1	3	τ'	τ	0	-1
$T_{2g} = \chi_3$	3	τ'	τ	0	-1	3	τ	τ'	0	-1
$G_g = \chi_4$	4	-1	-1	1	0	4	-1	-1	1	0
$H_g = \chi_5$	5	0	0	-1	1	5	0	0	-1	1
$A_u = \chi_6$	1	1	1	1	1	-1	-1	-1	-1	-1
$T_{1u} = \chi_7$	3	τ	τ'	0	1	-3	$-\tau'$	$-\tau$	0	1
$T_{2u} = \chi_8$	3	τ'	τ	0	-1	-3	$-\tau$	$-\tau'$	0	1
$G_u = \chi_9$	4	-1	-1	1	0	-4	1	1	-1	0
$H_u = \chi_{10}$	5	0	0	-1	1	-5	0	0	1	-1

Some notation here does double duty. For example, when the symbol i appears as one of the numbers in the character table, it represents the complex number whose square is -1. When it appears as the name of group element, it denotes *inversion*, which is a group operation which multiplies each coordinate by -1. (It corresponds to $-I$, the negative of the identity matrix.) The symbols T (with subscripts), when they name a particular representation, are not to be confused with our labeling of tetrahedra. The symbols C_n denote rotation of $2\pi/n$ about a line in 3-space as well as cyclic groups of order n. The symbol σ is used to signify reflection about a plane in 3-space, and it should not be confused with C_2, which is not a reflection, although both have order 2 as group elements. The symbol S with subscripts corresponds to reflection about a plane, followed by a rotation about a line perpendicular to that plane.

Each of these groups $\mathcal{G} \times \mathcal{S}_2$ has twice the number of elements as $\mathcal{G}$, twice the number of conjugacy classes as $\mathcal{G}$, and twice the number of irreducible representations as $\mathcal{G}$. For each conjugacy class say $[g]$ of $\mathcal{G}$, $[(g,1)]$ and $[(g,-1)]$ are distinct conjugacy classes of $\mathcal{G} \times \mathcal{S}_2$. For each representation ρ of $\mathcal{G}$ and g in $\mathcal{G}$, $\rho_1((g,1)) = \rho(g), \rho_1((g,-1)) = \rho(g)$, and $\rho_2((g,1)) = \rho(g), \rho_2((g,-1)) = -\rho(g)$ defines two other corresponding irreducible representations of $\mathcal{G} \times \mathcal{S}_2$. In the enumeration generators of the elements of $\mathcal{G}$, we can add one extra element, say $g_0 = -1$. Then $g_0^a g_1^b, \dots$, where $a = 0, 1, b = \dots$, enumerate $\mathcal{G}$.

This completes the description of our six basic groups.

11.7.3 Determinant plots.

We are now in a position to follow the method shown in Section 11.4 (illustrated by the example in Section 11.4.1) for the basic groups described above, as we have an explicit set of matrices that form each irreducible representation of each of our six groups.

Suppose we have chosen the group $\mathcal{G}$ and two group elements c_1 and c_2 in $\mathcal{G}$ that generate it (c_1 and c_2 will correspond to two transitivity classes of cables in the final tenesgrity we construct). This means that every element of $\mathcal{G}$ can be written as a product of some number of these two group elements in some order, and is necessary if we wish the final stress matrix that we construct to be positive semi-definite, given that the tensegrity is connected by struts and cables. We also choose s_1 in $\mathcal{G}$ not equal to any of $c_1, c_1^{-1}, c_2, c_2^{-1}$, which will correspond to the transitivity class of the strut.

From the definition in (11.4.2), for each irreducible representation ρ_i, we form the matrix $\mathbf{\Omega}_i = \mathbf{\Omega}_i(\omega_1, \omega_2, \omega_{-1})$ in terms of the real variables $\omega_1, \omega_2, \omega_{-1}$.

$$\mathbf{\Omega}_i = \omega_1 \mathbf{\Omega}_i(c_1) + \omega_2 \mathbf{\Omega}_i(c_2) + \omega_{-k} \mathbf{\Omega}_i(s_k).$$

When $\omega_1 > 0, \omega_2 > 0, \omega_{-1} \geq 0$ it is easy to see that $\mathbf{\Omega}_i$ is positive definite for ρ_i not the trivial representation. This is because the graph of the tensegrity that it defines is connected and effectively has only cables. So the only kernel vector comes from the trivial representation.

We now define the following, which we call the *i-th determinant polynomial* corresponding to the non-trivial irreducible representation ρ_i of $\mathcal{G}$,

$$\Delta_i(\omega_1, \omega_2, \omega_{-1}) = \det(\mathbf{\Omega}_i(\omega_1, \omega_2, \omega_{-1})).$$

Notice that Δ_i is implicitly a function of the choice of c_1, c_2 and s_1, and when $\omega_1 > 0, \omega_2 > 0, \omega_{-1} \geq 0$, then $\Delta_i(\omega_1, \omega_2, \omega_{-1}) > 0$ for all $i \neq 1$. Since we are ultimately interested in when $\omega_1 > 0, \omega_2 > 0$, we can normalize the force coefficients by dividing all the ω_i's by $\omega_1 + \omega_2$, and this has the effect that we may assume that $\omega_1 + \omega_2 = 1$. Thus we are essentially considering the two variable polynomial $\Delta_i(\omega_1, 1 - \omega_1, \omega_{-1})$.

Define the following region in (ω_1, ω_{-1}) space

$$R = R(c_1, c_2, s_1) = \{(\omega_1, \omega_{-1}) \mid 0 < \omega_1 < 1, \ \Delta_i(\omega_1, 1 - \omega_1, \omega_{-1}) > 0, \text{ for all } i \neq 1\}.$$

Each region R is convex, because the sum of positive definite matrices is positive definite. This means that if (ω_1, ω_{-1}) and $(\omega_1', \omega_{-1}')$ are in R, then so is any point on the line segment connecting them $((t\omega_1 + (1-t)\omega_1, t\omega_{-1}' + (1-t)\omega_{-1}')$, where $0 \leq t \leq 1)$.

Next fix $\omega_1 > 0$ and $\omega_2 = 1 - \omega_1 > 0$, and vary ω_{-1}. As $\omega_{-1} \to -\infty$ eventually $\mathbf{\Omega}$ must have a negative eigenvalue. So every vertical ray from the horizontal axis, starting in the interval from 0 to 1, must eventually leave R. So that point on the lower boundary of R corresponds to a force coefficient $\omega_{-1} < 0$ where, for some i, $\mathbf{\Omega}_i$ is singular, but still positive semi-definite (indeed, every point on the boundary of R corresponds to a positive semi-definite $\mathbf{\Omega}$).

The zero set of each of the polynomials Δ_i corresponds to values of the force coefficients where $\mathbf{\Omega}_i$ is singular, and the boundary of R must be part of that zero set. The i for which $\Delta_i(\omega_1, 1 - \omega_1, \omega_{-1}) = 0$ indicates which representation ρ_i corresponds to the singular $\mathbf{\Omega}_i(\omega_1, \omega_2, \omega_{-1})$. So the result of this analysis is that for fixed ω_1, ω_2, the critical force coefficient ω_{-1}, and the representation ρ_i to go with it, can be found by finding the smallest magnitude negative root of the polynomials $\Delta_i(\omega_1, \omega_2, \omega_{-1})$. We refer to that representation ρ_i as the *winner*.

Typical cases are shown later in figures 11.3 to 11.9. Each figure shows in (a) a perspective view of the resultant tensegrity, and in (b) the determinant plot showing the choice of stresses to generate the tensegrity by a cross, on the boundary of R. For each of the cases shown, the winning representation is 3-dimensional, giving a positive-definite stress matrix that is rank-deficient by 4. In each tensegrity, the cables are shown in red and blue (blue carrying the force coefficient ω_1, red carrying ω_2). Examples are shown for the groups $\mathcal{A}_4, \mathcal{A}_4 \times \mathcal{S}_2, \mathcal{S}_4, \mathcal{S}_4 \times \mathcal{S}_2, \mathcal{A}_5, \mathcal{A}_5 \times \mathcal{S}_2$.

For each figure 11.3 to 11.9, the choice of struts and cables, c_1, c_2, s_1, is given. However, to understand how those choices can be made, we need to define equivalence classes of tensegrity Cayley graphs, as described in the next section.

11.7.4 Cayley graphs.

We have seen that if we are given the elements c_1, c_2, s_1 in one of our groups $\mathcal{G}$, it is possible determine those geometric representations of a tensegrity, where c_1 and c_2 correspond to transitivity classes of cables, and s_1 corresponds to a transitivity class of struts. In principle, we could enumerate all such triples of elements of $\mathcal{G}$ and do the calculations for each of them, but that would involve several redundant cases, since several pairs of such tensegrities would be essentially identical. With this in mind, for any finite group $\mathcal{G}$ we define the *tensegrity Cayley graph* $\Gamma = \Gamma(c_1, \dots, s_1, \dots)$ corresponding to any finite number of elements $c_1, c_2, \dots$ and $s_1, \dots$ of $\mathcal{G}$ as follows. The vertices of Γ are the elements of $\mathcal{G}$. For any pair of elements g_1, g_2 of $\mathcal{G}$, there is an unoriented edge between them, labeled as a cable, if there is a c_i such that $g_1 c_i = g_2$ or $g_2 c_i = g_1$. Similarly, for any pair of elements g_1, g_2 of $\mathcal{G}$, there is an unoriented edge between them, labeled as a strut, if there is a s_i such that $g_1 s_i = g_2$ or $g_2 s_i = g_1$.

The standard definition of a Cayley graph does not usually make a distinction between cables and struts, but this is natural for us. This can be regarded as a sort of coloring of the edges of the standard Cayley graph.

An isomorphism $\alpha : \mathcal{G} \to \mathcal{G}$ of $\mathcal{G}$ to itself is called an *automophism*. For example, for any g_0 in $\mathcal{G}$ the function $g \to g_0 g g_0^{-1}$, which is conjugation by g_0, is automorphism of $\mathcal{G}$ called an *inner automorphism*. If an automorphism of a group is not an inner automorphism, it is called an *outer automorphism*.

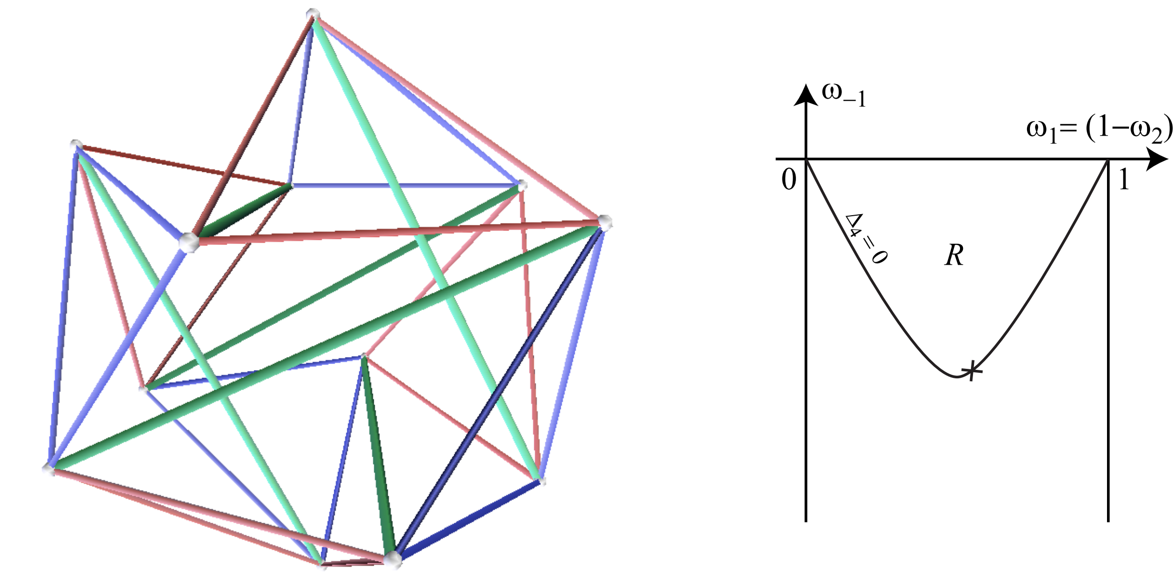
The following are some easy consequences of our definitions.

i.) If any cable or strut is replaced by its inverse, the corresponding tensegrity Cayley graph is the same.

ii.) If $\alpha : \mathcal{G} \to \mathcal{G}$ is an automorphism of $\mathcal{G}$, then the tensegrity Cayley graph $\Gamma(c_1, \dots, s_1, \dots)$ is the same as the tensegrity Cayley graph of $\Gamma(\alpha(c_1), \dots, \alpha(s_1), \dots)$.

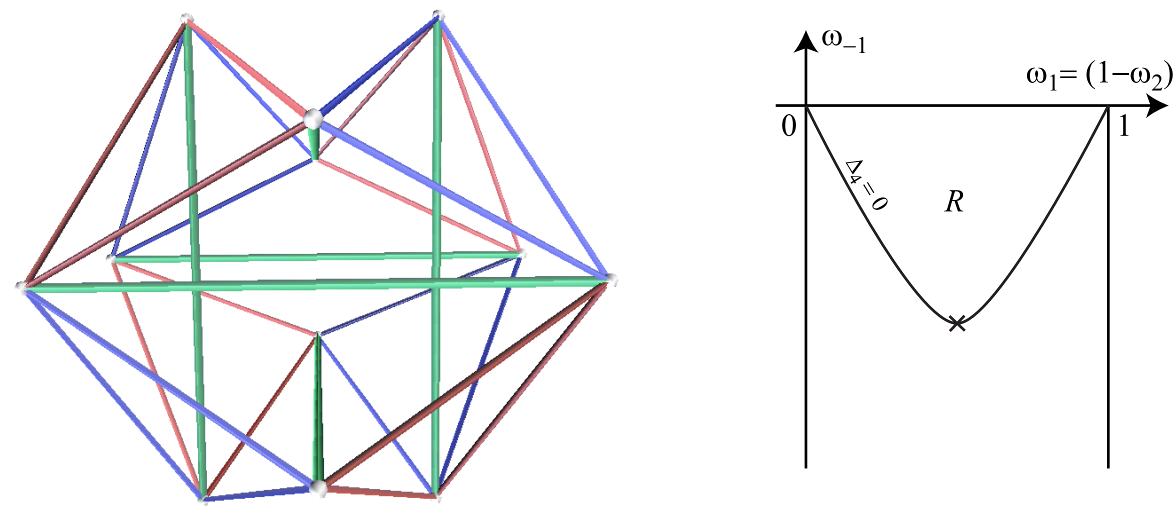
iii.) The elements $c_1, c_2, \dots$ generate $\mathcal{G}$ if and only if the cable subgraph of Γ is connected.

We can regard an automorphism of the group $\mathcal{G}$ as a way of relabling the vertices of the associated Cayley graph Γ that is consistent with the group structure. The group $\mathcal{G}$ acts on Γ as a group of symmetries in the sense that left multiplication by any element of $\mathcal{G}$ takes Γ to itself.



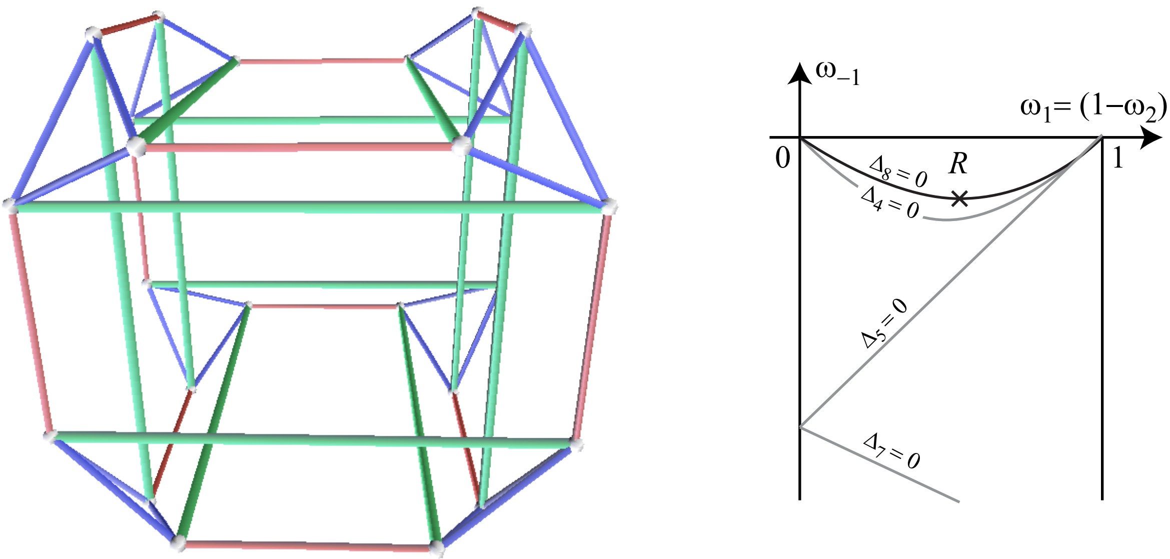
(a) (b)

Figure 11.3: An $\mathcal{A}_4$ tensegrity with cables $\{(134),(234)\}$ and strut$\{(14)(23)\}$ — in this case the force coefficient is unequal in the two classes of cables. (a) The physical configuration. (b) The determinant plot, showing the choice of stresses by a cross, on the boundary of the region R, which ensures that the stress matrix is positive semi-definite. In this case, ρ_4 is the winner, which is a faithful representation. As ρ_4 is 3-dimensional representation, the stress matrix is rank-deficient by 4.



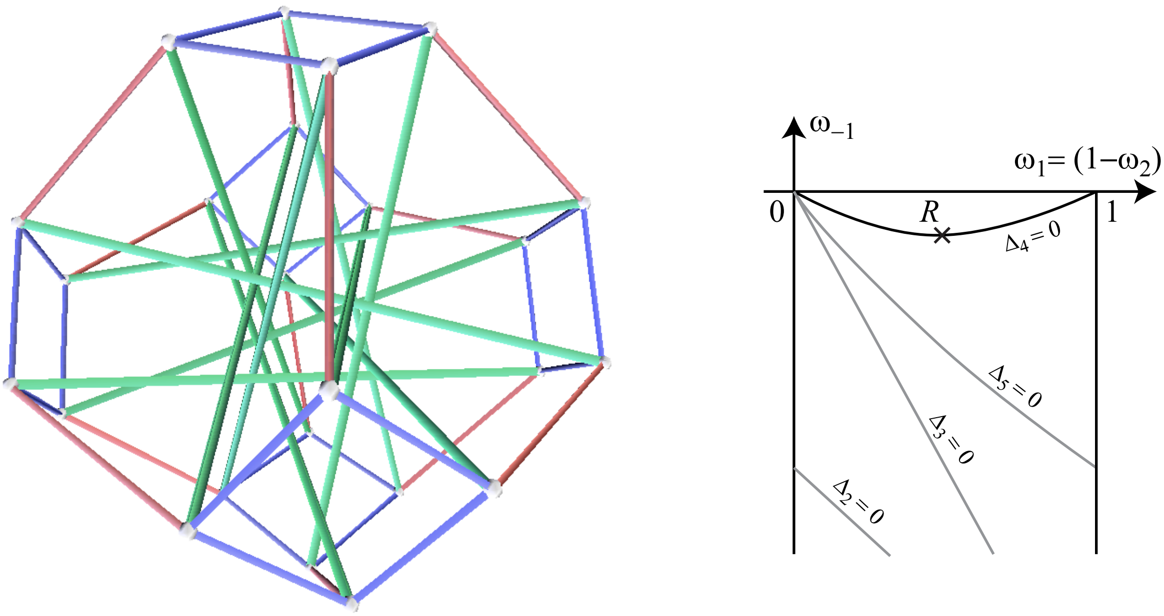
(a) (b)

Figure 11.4: The same choice of group, cables and struts as in figure 11.3, but now with the force coefficient in the cables equal, which creates an additional reflection symmetry. This tensegrity is used as a baby toy called the "Sqwish".



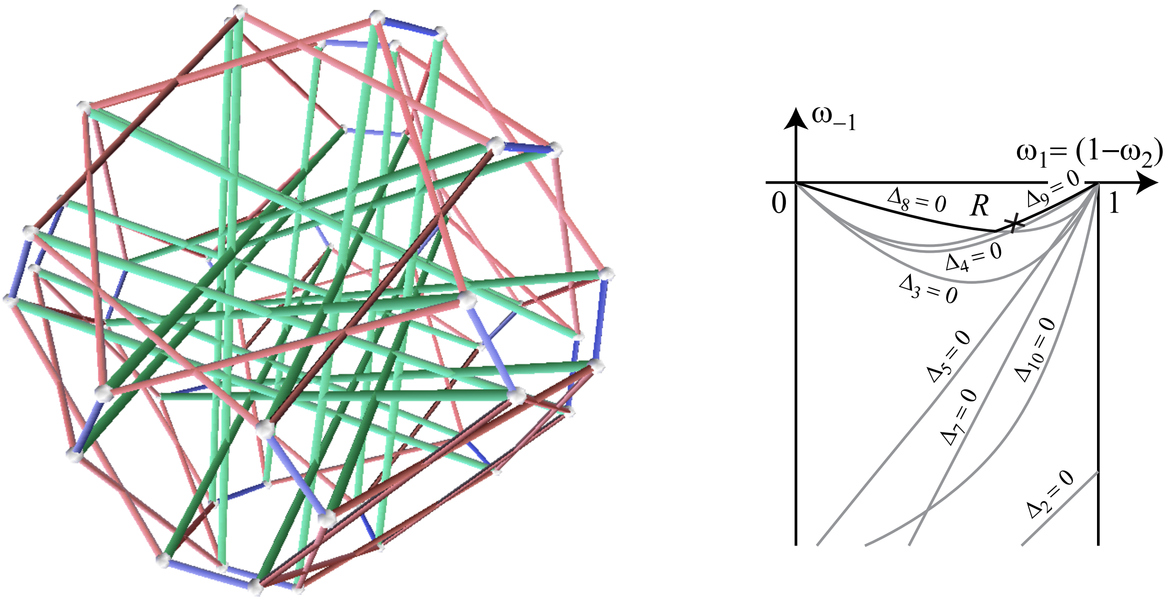
(a) (b)

Figure 11.5: An $\mathcal{A}_4 \times \mathcal{S}_2$ tensegrity with cables $\{(124), -(13)(24)\}$ and strut $\{-(12)(34)\}$ and its determinant plot. In this case, ρ_8 is the winner, which is a faithful representation. As ρ_8 is 3-dimensional representation, the stress matrix is rank-deficient by 4.



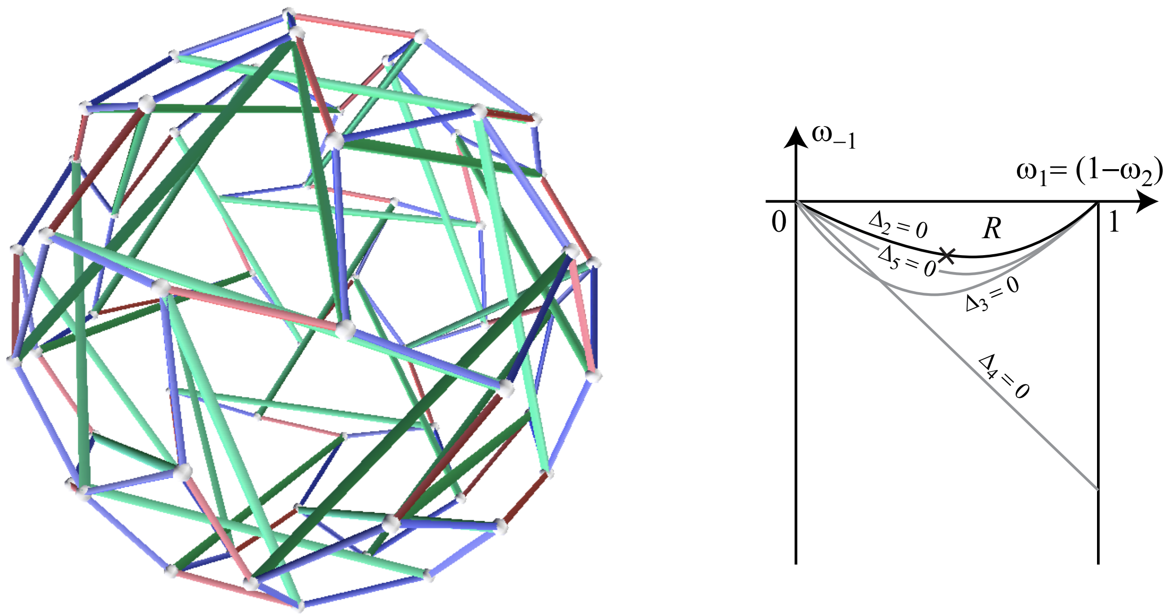
(a) (b)

Figure 11.6: An $\mathcal{S}_4$ tensegrity with cables $\{(1423), (13)\}$ and strut$\{(12)\}$ and its determinant plot. In this case, ρ_4 (faithful, 3-dimensional) is the winner.



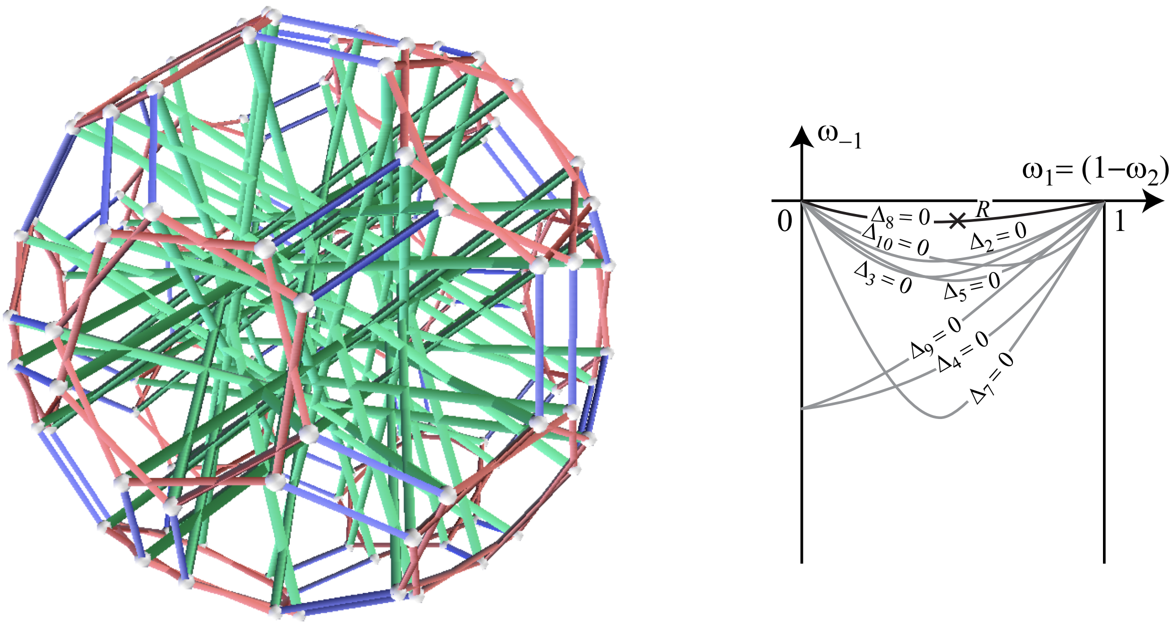
(a) (b)

Figure 11.7: An $\mathcal{S}_4 \times \mathcal{S}_2$ tensegrity with cables $\{-(14), (1234)\}$ and strut $\{(34)\}$ and its determinant plot. In this case, ρ_9 (faithful, 3-dimensional) is the winner. Note that in this case, the force coefficient ω_1 (in the blue cables) has to be sufficient to ensure that the 3-dimensional ρ_9 is the winner — ρ_8 is 2-dimensional, and hence does not give a 3-dimensional configuration.



(a) (b)

Figure 11.8: An $\mathcal{A}_5$ tensegrity with cables $\{(15243), (15)(34)\}$ and strut $\{(13)(24)\}$, together with its determinant plot. The cable graph is that of the soccer ball, a truncated icosahedron, although the hexagons are distorted. In this case, ρ_2 (faithful, 3-dimensional) is the winner.



(a) (b)

Figure 11.9: An $\mathcal{A}_5 \times \mathcal{S}_2$ tensegrity with cables $\{-(15)(24), (14532)\}$ and strut $\{(13)(24)\}$ and its determinant plot. In this case, ρ_8 (faithful, 3-dimensional) is the winner. In this case, as the force coefficient ω_2 (in the red cables) is increased, the red cables shorten and the configuration moves towards a regular icosahedron where sets of 10 vertices converge.

For Cayley graphs, it is usual to assume that the elements that are used to define it actually generate it, so the associated Cayley graph is connected. For us, it will be natural, also, to assume that the subgraph of Γ, determined by the edges labled cables, be connected.

11.7.5 Automorphisms

In light of the discussion in the previous subsection, we will determine all the automorphisms of the dihedral groups and the six groups that we have considered. One other definition is useful. We say that a subgroup $\mathcal{H}$ of the group $\mathcal{G}$ is *normal* (or equivalently *self-conjugate*) if for all g in $\mathcal{G}$, $g\mathcal{H}g^{-1} = \mathcal{H}$, where $g\mathcal{H}g^{-1} = \{ghg^{-1} \mid h \text{ in } \mathcal{H}\}$. For example, the group of even permutations of n symbols $\mathcal{A}_n$ is a normal subgroup of $\mathcal{S}_n$ the group of all permutations of n symbols, for $n = 2, 3, \dots$. It is a well-known fact, which can be read off from the character table, that $\mathcal{A}_5$ has no no normal subgroups other than itself and the trivial group consisting of just the identity element.

If a group $\mathcal{H}$ is a normal subgroup of a larger group $\mathcal{G}$, then it is easy to see that conjugation of $\mathcal{H}$ by any element of $\mathcal{G}$, $h \to ghg^{-1}$ is an automorphism of $\mathcal{H}$. So for the groups that we considered, $\mathcal{A}_4$ and $\mathcal{A}_5$ are normal subgroups of $\mathcal{S}_4$ and $\mathcal{S}_5$, respectively, and so conjugation by elements in the larger symmetric group provides outer automorphisms.

Conversely, suppose that we identify $\mathcal{G}$ as a subgroup of $GL(n, \mathbb{R})$, the group of all n-by-n non-singular matrices, for some $n = 1, 2, \dots$. We can regard the identity map as a representative of one class of equivalent representations. Then any automorphism α of $\mathcal{G}$ is also a representation of $\mathcal{G}$ of the same dimension. Of course, if α is obtained by conjugation by an element of $\mathcal{G}$ (an inner automorphism) or even conjugation by an element of $GL(n, \mathbb{R})$, α will be equivalent as a representation to the identity representation. But if there is an automorphism that does not arise from conjugation by an element of $GL(n, \mathbb{R})$, then it will determine an inequivalent representation and this can be determined in the character table for $\mathcal{G}$.

Since we are dealing largely with permutation groups, it is helpful to describe the operation of conjugation by an element of S_n. We wish to describe $\alpha(g) = g_0 g g_0^{-1}$. First regard each of these elements as functions, i.e. permutations, of the symbols $\{1, \dots, n\}$. Then $\alpha(g)(g_0(i)) = g_0(g(i))$, and so if we write $g = (a, b, c, \dots)$ in disjoint cycle notation, then $\alpha(g) = (g_0(a), g_0(b), \dots)$ in disjoint cycle notation. For example, if $g_0 = (12)$ and $g = (12345)$ in disjoint cycle notation, then $\alpha(g) = (21345)$ in disjoint cycle notation. This is useful to simplify some of the calculations.

We now describe the automorphisms of the dihedral groups and our six other groups.

$\mathcal{D}_n$, n odd, $n \geq 3$.

We can describe any automorphism α of $\mathcal{D}_n$ by the value of $\alpha(r)$ and $\alpha(s)$. Since the order of r is n, $\alpha(r)$ must also have order n. But the only possibilities are $\alpha(r) = r^j$, where j relatively prime to n. Similarly, since the order of s is 2, $\alpha(s) = r^k s$, for some $k = 1, \dots n$. It is easy to check that r^j and $r^k s$ generate $\mathcal{D}_n$, and that $(r^k s) r^j (r^k s) = (r^j)^{-1}$. Thus such any such α for j relatively prime to n determines an automorphism of $\mathcal{D}_n$.

$\mathcal{D}_n$, n even $n \geq 4$.

Again we can describe any automorphism α of $\mathcal{D}_n$ by the value of $\alpha(r)$ and $\alpha(s)$. Since r has order n, $\alpha(r)$ must also have order n, which implies $\alpha(r) = r^j$, where j is relatively prime to n as before. But now there are more elements that have order 2 that could be images of s. The element $r^{n/2}$ cannot be the image of s, since r^j and $r^{n/2}$ do not generate all of $\mathcal{D}_n$. On the other hand $\alpha(s) = r^k s$ for any $k = 1, \ldots,$ and $\alpha(r) = r^j$, j relatively prime to n does define an automorphism of $\mathcal{D}_n$ as before.

$\mathcal{A}_4$.

Recall that the irreducible 3-dimensional representation of $\mathcal{A}_4$ assigns the linear extension of the corresponding permutation of the vertices of a regular tetrahedron centered at the origin. Any automorphism of the image of this representation in $GL(3, \mathbb{R})$ arising from the conjugation by an element of $GL(3, \mathbb{R})$, that is a matrix with positive determinant, will be an inner automorphism. If the matrix has a negative determinant, it will correspond to conjugation by an element in $\mathcal{S}_4$, since the elements of $\mathcal{S}_4$ correspond to arbitrary permutations of the vertices of the regular tetrahedron. Thus conjugation $\mathcal{A}_4$ by elements in the larger group $\mathcal{S}_4$ describe all the automorphisms of $\mathcal{A}_4$.

$\mathcal{S}_4$.

There are two distinct 3-dimensional irreducible representations of $\mathcal{S}_4$, but their images in $Gl(3, \mathbb{R})$ are distinct. One image contains some matrices with negative determinant, and the other only those with positive determinant. So there are only inner automorphisms of $\mathcal{S}_4$.

$\mathcal{A}_5$.

This is similar to $\mathcal{A}_4$ in that the only automorphisms are those coming from conjugation by elements in the larger group S_5. This can be seen by considering the irreducible 4-dimensional representation, or if one looks at the two irreducible 3-dimensional representations, their images can be taken to be the rotations of the regular dodecahedron (or equivalently the icosahedron). In addition to the inner automorphisms of $\mathcal{A}_5$, conjugation by an odd permutation of S_5 (as described above), takes an element of order 5 in one conjugacy class to the other conjugacy class, and this permutes the two irreducible 3-dimensional representations.

$\mathcal{A}_4 \times \mathcal{S}_2$.

The group of symmetries of a cube that induce an even permutation on the long diagonals of the cube is the image of the only 3-dimensional irreducible representation of $\mathcal{A}_4 \times \mathcal{S}_2$. So the only automorphisms of $\mathcal{A}_4 \times \mathcal{S}_2$ are restrictions of symmetries of cube. These are then the automorphisms of the group $\mathcal{A}_4$ (conjugation by an element of $\mathcal{S}_4$ not changing the sign of $\mathcal{S}_2$. In other words $\alpha(g, z) = (g_0 g g_0^{-1}, z)$, where g is in $\mathcal{A}_4$, g_0 is in $\mathcal{S}_4$, and z is in $\mathcal{S}_2$.

Note that here and later we regard the group $\mathcal{S}_2 = \{1, -1\}$ where 1 is the identity element, and -1 is the other element.

$\mathcal{S}_4 \times \mathcal{S}_2$.

This group is isomorphic to the full group of symmetries of the cube. There are two inequivalent irreducible one-to-one 3-dimensional representations of this group as can be seen from the character table. The automorphism that takes one representation to the other is given by $\alpha(g,z) = (g_0 g g_0^{-1}, \theta(g)z)$, where g is in $\mathcal{S}_4$, g_0 is in $\mathcal{S}_4$, z is in $\mathcal{S}_2$ and $\theta : \mathcal{S}_4 \to \mathcal{S}_2$ is the group homomorphism that assigns $+1$ to an even permutation and -1 to an odd permutation. So all automorphisms of $\mathcal{S}_4 \times \mathcal{S}_2$ are described as inner automorphisms or one of the automorphisms above. For example, the pair of elements $-(123), (1234)$ are taken to the corresponding pair $-(213), -(2134)$ by an automorphism of the first type.

$\mathcal{A}_5 \times \mathcal{S}_2$.

This group is isomorphic to the full group of symmetries of the regular dodecahedron (or icosahedron). But again this group is the image of two distinct irreducible one-to-one 3-dimensional representations. But then it is easy to see that the only automorphisms are given as the product of conjugation on the $\mathcal{A}_5$ factor by an element of S_5 and the identity on the $\mathcal{S}_2$ factor.

11.7.6 Enumeration of tensegrity Cayley graphs.

From the discusion in Section 11.7.4 we see what the conditions are for the group elements c_1, c_2, s_1 to define the same tensegrity Cayley graph $\Gamma(c_1, c_2, s_1)$. With this in mind we define the following set, which we call the *defining set*.

$$D = D(c_1, c_2, s_1) = (\{\{c_1, c_1^{-1}\}, \{c_2, c_2^{-1}\}\}, \{s_1, s_1^{-1}\}).$$

Here we are using set notation, where $\{x,y\} = \{y,x\}$, but $(x,y) \neq (y,x)$, unless $x = y$. We only create a defining set D when all three of the sets $\{c_1, c_1^{-1}\}, \{c_2, c_2^{-1}\}, \{s_1, s_1^{-1}\}$ are distinct. With this notation, we see that D remains the same if any of the elements are replaced by their inverse, or the roles of c_1 and c_2 are reversed. Note, also, that if an element is replaced by its inverse, the set it defines collapses to a singleton. From the discussion to this point we have the following.

Proposition 11.7.1. ***Cayley defining sets.*** *For a group $\mathcal{G}$, two tensegrity Cayley graphs $\Gamma(c_1, c_2, s_1)$ and $\Gamma(c_1', c_2', s_1')$ are the same if and only if there is an automorphism α of $\mathcal{G}$ such that $D(\alpha(c_1), \alpha(c_2), \alpha(s_1)) = D(c_1', c_2', s_1')$.*

We also insist that c_1 and c_2 generate $\mathcal{G}$ in order that the cable graph be connected, which in turn is needed if there is any chance for the tensegrity to be super-stable.

Our catalogue is then organized so that for each group $\mathcal{G}$ there is one entry for each equivalence class of triples of elements in $\mathcal{G}$, where two triples are equivalent if they define the same defining set. Furthermore it is natural to collect those triples together that define the same cable graph. For each pair of elements c_1 and $c2$ in $\mathcal{G}$, define the *cable defining set* as

$$C = C(c_1, c_2) = \{\{c_1, c_1^{-1}\}, \{c_2, c_2^{-1}\}\}.$$

Then we first consider the C-equivalence classes, where (c_1, c_2) is C-equivalent to (c_1', c_2') if there is an automorphism α of $\mathcal{G}$ such that $C(\alpha(c_1), \alpha(c_2)) = C(c_1', c_2')$. Then for each of these C-equivalence classes, there will be several equivalence classes of triples, each defining a distinct tensegrity Cayley graph. Keep in mind, though, that c_1 and c_2 must generate $\mathcal{G}$.

For example, for the group $\mathcal{A}_4$ there are only two C-equivalence classes, which are represented by $\{(134),(243)\}$ and $\{(124),(14)(23)\}$. This amounts to saying that any two generators of $\mathcal{A}_4$ that are of order 3 are C-equivalent, and any two generators, one of order 3, the other of order 2, are C-equivalent. Furthermore, these are the only pairs of generators of $\mathcal{A}_4$. Each of these C-equivalence classes has 3 tensegrity Cayley graphs, making 6 in all.

For the dihedral groups, and from the discussion in Section 11.7.5, we see that the only C-equivalence class for $\mathcal{D}_n, n = 3,4,\ldots$ is represented by $\{r,s\}$, since any two elements that generate $\mathcal{D}_n$ are an automorphic image of r and s.

Table 11.1 gives the C-equivalence classes for the six non-dihedral groups we have considered.

11.7.7 Algorithm.

We have all the tools to describe the process that goes into creating the catalogue of symmetric tensegrities. The goal and restrictions are as follows.

i.) The tensegrity has a symmetry group $\mathcal{G}$ isomorphic to one of $\mathcal{D}_n, n = 3,4,\ldots$, $\mathcal{A}_4$, $\mathcal{S}_4$, $\mathcal{A}_5$, $\mathcal{A}_4 \times \mathcal{S}_2$, $\mathcal{S}_4 \times \mathcal{S}_2$, or $\mathcal{A}_5 \times \mathcal{S}_2$.

ii.) The group $\mathcal{G}$ acts transitively on the vertices of the tensegrity.

iii.) The group $\mathcal{G}$ acts freely on the vertices of the tensegrity. That is the only group element that fixes any vertex is the identity.

iv.) The tensegrity has only two transitivity classes of cables, and one transitivity class of strut.

With the above in mind, our goal is to display all those tensegrities that satisfy the conditions above that have a positive semi-definite stress matrix with a 4-dimensional kernel. There is choice of the ratio of the lengths of cables from the two transitivity classes, for example. For many of our pictures, we have arbitrarily decided to make the two cable force coefficient equal to each other.

Our method in creating the catalogue of pictures of the final tensegrities is as follows.

i.) List all the equivalence classes of tensegrity Cayley graphs as described in Section 11.7.6.

ii.) For each class, calculate the determinant polynomials for each of the irreducible representations of $\mathcal{G}$.

iii.) Calculate the winning representation, say j as described in Section 11.7.3.

iv.) If the winning representation is 3-dimensional, and the stresses that correspond to the representation are $\omega_1, \omega_2, \omega_{-1}$, compute the 3-by-3 matrix $\mathbf{\Omega}_j = \mathbf{\Omega}_j(\omega_1, \omega_2, \omega_{-1})$.

Group	$\mathcal{A}_4$	$\mathcal{S}_4$	$\mathcal{A}_5$
C-equivalence classes	$\{(134),(243)\}^*$ $\{(124),(14)(23)\}^*$	$\{(1423),(234)\}^*$ $\{(12),(143)\}^*$ $\{(1423),(13)\}^*$ $\{(1423),(1324)\}$	$\{(134),(23)(45)\}^*$ $\{(15)(24),(14532)\}$ $\{(12354),(143)\}^*$ $\{(12354),(145)\}$ $\{(12354),(14235)\}$ $\{(15243),(15)(34)\}^*$ $\{(142),(354)\}$ $\{(14325),(14253)\}$

Group	$\mathcal{A}_4 \times \mathcal{S}_2$	$\mathcal{S}_4 \times \mathcal{S}_2$	$\mathcal{A}_5 \times \mathcal{S}_2$
C-equivalence classes	$\{(124),-(13)(24)\}^*$ $\{(142),-(134)\}$ $\{-(243),(12)(34)\}$ $\{-(234),-(12)(34)\}$ $\{-(234),-(134)\}$	$\{-(1342),(34)\}$ $\{-(132),-(14)\}$ $\{-(1423),(1234)\}$ $\{-(1423),-(142)\}$	$\{-(13425),(23)(45)\}$ $\{(23)(45),-(152)\}$ $\{-(13425),-(23)(45)\}$ $\{(124),-(23)(45)\}$ $\{-(23)(45),(15342)\}$ $\{-(23)(45),-(13245)\}$ $\{(23)(45),-(14523)\}$ $\{(15243),-(142)\}$ $\{-(14)(25),(13524)\}$ $\{-(132),-(154)\}$ $\{-(132),(12354)\}$ $\{-(25)(34),-(123)\}$ $\{(354),-(123)\}$ $\{-(12543),-(123)\}$ $\{-(15342),(153)\}$ $\{-(13524),-(15342)\}$ $\{-(125),-(13524)\}$ $\{-(14253),(13254)\}$ $\{-(15243),-(14253)\}$ $\{(12435),-(13524)\}$ $\{-(12435),(253)\}$

Table 11.1: C-equivalence classes for the six non-dihedral groups we have considered. Bear in mind that for each choice of cable graph, there are several choices of struts that give distinct tensegrity Cayley graphs. The symbol $\{\}^*$ indicates that the cable Cayley graph is planar. It turns out that this implies that the representation is almost always determined by the cable graph only.

v.) By construction $\mathbf{\Omega}_j$ is singular, and so has a non-zero (column) vector p_1 such that $\mathbf{\Omega}_j p_1 = 0$. Then the configuration for the desired tensegrity is given by

$$(\rho_j(g_1)p_1, \rho_j(g_2)p_1, \dots, \rho_j(g_n)p_1)$$

for all the elements $g_1, \dots, g_n$ of $\mathcal{G}$. The rule for determining which pairs of elements determine cables or struts is given by the rule in Section 11.7.4.

11.7.8 Results for the dihedral groups.

We can give a complete description of the tensegrities for the conditions in Subsection 11.7.7.

We know from Subsection 11.7.5 and Subsection 11.7.6 that the only C-equivalence class for $\mathcal{D}_n, n \geq 3$, is $\{r, s\}$. However, we must make a choice of a group element s_1 corresponding to a strut. It turns out that if $s_1 = r^j$ for some j, then the resulting critical configuration will be either 2-dimensional or it will be 4-dimensional. So we will concentrate on the case when $s_1 = r^j s$, for $j = 2, \dots, n-1$. We calculate local stress matrices for each representation, ψ_j, for $j = 1, \dots$.

$$\begin{aligned}
\mathbf{\Omega}_j &= \omega_1(\rho_j(1-r^k) + \rho_j(1-r^k)) + \omega_2\rho_j(1-s) + \omega_{-1}(1-r^k s) \\
&= \omega_1 \begin{bmatrix} 2(1-\cos(\frac{2\pi j}{n})) & 0 \\ 0 & 2(1-\cos(\frac{2\pi j}{n})) \end{bmatrix} + \omega_2 \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \omega_{-1} \begin{bmatrix} 1-\cos(\frac{2\pi jk}{n}) & -\sin(\frac{2\pi jk}{n}) \\ -\sin(\frac{2\pi jk}{n}) & 1+\cos(\frac{2\pi jk}{n}) \end{bmatrix} \\
&= \begin{bmatrix} 2(1-\cos(\frac{2\pi j}{n}))\omega_1 + (1-\cos(\frac{2\pi jk}{n}))\omega_{-1} & -\sin(\frac{2\pi jk}{n})\omega_{-1} \\ -\sin(\frac{2\pi jk}{n})\omega_{-1} & 2(1-\cos(\frac{2\pi j}{n}))\omega_1 + 2\omega_2 + (1+\cos(\frac{2\pi jk}{n}))\omega_{-1} \end{bmatrix}
\end{aligned}$$

The local stress matrix for the representation ρ_2 is $[\omega_2 + \omega_{-1}]$, and so its determinant polynomial is $\omega_2 + \omega_{-1}$. This one-dimensional representation must contribute to the kernel of $\mathbf{\Omega}$, if we want to have a three-dimensional super stable tensegrity. So we assume that $\omega_2 + \omega_{-1} = 0$ and thus $\omega_2 = -\omega_{-1}$. Then

$$\mathbf{\Omega}_j = \begin{bmatrix} 2(1-\cos(\frac{2\pi j}{n}))\omega_1 + (1-\cos(\frac{2\pi jk}{n}))\omega_{-1} & -\sin(\frac{2\pi jk}{n})\omega_{-1} \\ -\sin(\frac{2\pi jk}{n})\omega_{-1} & 2(1-\cos(\frac{2\pi j}{n}))\omega_1 - (1-\cos(\frac{2\pi jk}{n}))\omega_{-1} \end{bmatrix}.$$

So the corresponding determinant polynomial for the representation ψ_j is

$$\begin{aligned}
\Delta_j &= 4(1-\cos(\frac{2\pi j}{n}))^2\omega_1^2 - (1-\cos(\frac{2\pi jk}{n}))^2\omega_{-1}^2 - \sin(\frac{2\pi jk}{n})^2\omega_{-1}^2 \\
&= 4(1-\cos(\frac{2\pi j}{n}))^2\omega_1^2 - 2(1-\cos(\frac{2\pi jk}{n}))\omega_{-1}^2.
\end{aligned}$$

The critical ratio of force coefficients that will imply that $\Delta_j = 0$ is when

$$(\frac{\omega_1}{\omega_{-1}})^2 = 2\frac{(1-\cos(\frac{2\pi j}{n}))^2}{(1-\cos(\frac{2\pi jk}{n}))} = 4\frac{\sin(\frac{\pi j}{n})^4}{\sin(\frac{\pi jk}{n})^2}.$$

Equivalently,

$$\left|\frac{\omega_1}{\omega_{-1}}\right| = 2\frac{\sin(\frac{\pi j}{n})^2}{|\sin(\frac{\pi jk}{n})|}. \tag{11.7.1}$$

The following Lemma from Connelly and Terrell (1995) finishes this classification for dihedral groups.

Lemma 11.7.2. *For fixed n and $k = 1 \dots n-1$, the minimum value of $\frac{\sin(\frac{\pi j}{n})^2}{|\sin(\frac{\pi jk}{n})|}$ for $j = 1 \dots n-1$ occurs only when $j = 1$ or $j = n-1$.*

Proof. The statement of the Lemma is equivalent to the following inequality for $j = 2, \dots, n-2$.

$$\left|\frac{\sin(\frac{\pi jk}{n})}{\sin(\frac{\pi k}{n})}\right| < \frac{\sin(\frac{\pi j}{n})^2}{\sin(\frac{\pi}{n})^2}, \tag{11.7.2}$$

and since $\frac{\sin(\frac{\pi jk}{n})}{\sin(\frac{\pi k}{n})} > 1$, the inequality (11.7.2) follows from the inequality

$$\left|\frac{\sin(\frac{\pi jk}{n})}{\sin(\frac{\pi k}{n})}\right| \leq \left|\frac{\sin(\frac{\pi j}{n})}{\sin(\frac{\pi}{n})}\right|, \tag{11.7.3}$$

for $j = 2, \dots, n-2$. By replacing j by $n-j$ and k by $n-k$ when necessary, it is enough to show Equation (11.7.3) when $2 \leq j \leq n/2$, and $1 \leq k \leq n/2$. Using the fact that $0 \leq t \leq \pi/2$ implies that $2t/\pi \leq \sin t \leq t$, we see that

$$\left|\frac{\sin jk\frac{\pi}{n}}{\sin k\frac{\pi}{n}}\right| \leq \frac{1}{\frac{2k}{\pi n}\pi} = \frac{n}{2k} \text{ and } \frac{2j}{\pi} = \frac{\frac{2}{\pi}j\frac{\pi}{n}}{\frac{\pi}{n}} \leq \frac{\sin j\frac{\pi}{n}}{\sin\frac{\pi}{n}}.$$

Thus Equation (11.7.3) holds when $n/2k \leq 2j/\pi$, in other words when $\pi n/4 \leq jk$.

For the remaining cases $jk < \pi n/4 < n$, write $\sin t = t\,\prod_{m=1}^{\infty}(1 - \frac{t^2}{m^2\pi^2})$, which converges absolutely for all real t. Thus

$$\frac{\sin j\frac{\pi}{n}}{\sin\frac{\pi}{n}} = \frac{\frac{j}{n}\pi\prod_{m=1}^{\infty}(1-\frac{j^2}{n^2m^2})}{\frac{1}{n}\pi\prod_{m=1}^{\infty}(1-\frac{1}{n^2m^2})} = j\prod_{m=1, j\nmid m}^{\infty}\left(1-\frac{j^2}{n^2m^2}\right)$$

and

$$\frac{\sin jk\frac{\pi}{n}}{\sin\frac{k\pi}{n}} = \frac{\frac{jk}{n}\pi\prod_{m=1}^{\infty}(1-\frac{j^2k^2}{n^2m^2})}{\frac{k}{n}\pi\prod_{m=1}^{\infty}(1-\frac{k^2}{n^2m^2})} = j\prod_{m=1, j\nmid m}^{\infty}\left(1-\frac{k^2j^2}{n^2m^2}\right).$$

The quotients are indexed over the positive integers m which are not divisible by j, i.e. $j \nmid m$. When $jk < n$, it follows that for each m,

$$0 < \left(1-\frac{k^2j^2}{n^2m^2}\right) < \left(1-\frac{k^2}{n^2m^2}\right).$$

This implies Equation (11.7.3). □

This representation corresponds to the tensegrity where the c_1 cables form the edges of two convex polygons in parallel planes. An example is shown in Figure 1.2. The parameter k refers to the number of steps between the end of the lateral cable and strut that are adjacent to the same vertex. This analysis shows that for all values of $k = 1, \dots, n-1$, the corresponding stress matrix $\mathbf{\Omega}$ has 4 zero eigenvalues, with all the rest positive. It is easy to show that there are no affine motions preserving all the stressed directions. So these tensegrity structures are prestress stable, and super stable.

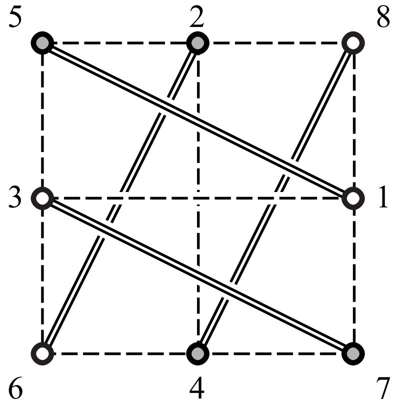


Figure 11.10: A tensegrity with $\mathbb{Z}_4$ symmetry ($\mathcal{S}_\triangle$ in Schoenflies notation) in $\mathbb{R}^3$. The nodes lie in two planes parallel with the paper, with the shaded nodes below. Nodes $1, 2, 3, 4$ form one transitivity class of nodes; nodes $5, 6, 7, 8$ form the other.

11.8 Non-transitive examples.

The essential observation in the calculations in this chapter is that the stress matrix is a linear combination of permutation matrices on the nodes of the tensegrity. This is still true for non-transitive and non-free case. We do one example of that sort in this section. Here we use an example due to Grünbaum and Shephard (1975). This tensegrity has symmetry group, in Schoenflies notation, $\mathcal{S}_4$, which is isomorphic to $\mathcal{C}_4$, the cyclic group of order 4, but is generated by an improper rotation S_4 in $\mathbb{R}^3$ - however, to avoid the confusion with the permutation group, we will refer to the group as $\mathbb{Z}_4$. The action of $\mathbb{Z}_4$ is not transitive on the nodes. There are two transitivity classes. Figure 11.10 shows the tensegrity in a top view. Until you have a model in your hand, it is hard to believe that the crossings are as indicated. Note that all the struts are disjoint.

The permutation of the vertices in this representaton is such that the generator g corresponds to the permutation $g \to (1234)(5678)$ in disjoint cycle notation. This is given by the improper rotation by $90°$ on the vertices with those labels. More generally, consider the two-by-two matrix

$$\mathbf{\Omega}(g) = \begin{bmatrix} \omega_1 + \omega_2 + \omega_4 + \omega_3 - g^2\omega_3 & -g^{-1}\omega_1 - \omega_2 - g\omega_4 \\ -g\omega_1 - \omega_2 - g^{-1}\omega_4 & \omega_1 + \omega_2 + \omega_4 \end{bmatrix}, \tag{11.8.1}$$

where

$$\begin{aligned} \omega_1 &= \omega_{17} = \omega_{28} = \omega_{35} = \omega_{48} \\ \omega_2 &= \omega_{18} = \omega_{25} = \omega_{36} = \omega_{47} \\ \omega_3 &= \omega_{13} = \omega_{24} \\ \omega_4 &= \omega_{15} = \omega_{26} = \omega_{37} = \omega_{48}, \end{aligned}$$

we consider each entry of $\mathbf{\Omega}(g)$ to be in a $\mathcal{G}$-algebra, and it is understood that an entry without a $\mathcal{G}$ coefficient implicitly is multiplied by the identity group operation.

Let ρ_R to be the regular representation of the cyclic group of 4 elements $\mathbb{Z}_4$ as a set of 4-by-4 permutation matrices. Then the crucial observation is that $\mathbf{\Omega}(\rho_R(g))$ can be identified as the standard stress matrix for the tensegrity graph of Figure 11.10, and that $\mathbf{\Omega}(\rho_j(g))$ are its components, where ρ_j, for $j = 1, \dots, 4$, are the irreducible representations of ρ_R. Since $\mathbb{Z}_4$ is an abelian group, all its representations are linear, but into the field of complex numbers. The description of $\mathbf{\Omega}(g)$ is obtained by choosing a particular representative node for each transitivity class, in this case node 1 and node 8. Each force coefficient ω_j contributes a term of the form $\omega_j(1 - g^k)$ to $\mathbf{\Omega}(g)$, where the identity term and the g term appear in the appropriate column and row of $\mathbf{\Omega}(g)$ depending on which pair of transitivity classes are being connected and which element g^k is needed to connect them using nodes 1 and 8 as base nodes.

The irreducible representations ρ_j of $\mathbb{Z}_4$ are such that $g \to 1, i, -1, -i$, for $j = 1, 2, 3, 4$, respectively. This gives the following by substituting into (11.8.1):

$$\mathbf{\Omega}(\rho_1) = \begin{bmatrix} \omega_1 + \omega_2 + \omega_4 & -\omega_1 - \omega_2 - \omega_4 \\ -\omega_1 - \omega_2 - \omega_4 & \omega_1 + \omega_2 + \omega_4 \end{bmatrix},$$

which has rank 1 and is positive semi-definite as long as $\omega_1 + \omega_2 + \omega_4 > 0$. For ρ_2 we get:

$$\mathbf{\Omega}(\rho_2) = \begin{bmatrix} \omega_1 + \omega_2 + 2\omega_3 + \omega_4 & i\omega_1 - \omega_2 - i\omega_4 \\ -i\omega_1 - \omega_2 + i\omega_4 & \omega_1 + \omega_2 + \omega_4 \end{bmatrix},$$

which has determinant $(\omega_1+\omega_2+\omega_4)^2 + 2(\omega_1+\omega_2+\omega_4)\omega_3 - (\omega_1-\omega_4)^2 - \omega_2^2$. Note that $\mathbf{\Omega}(\rho_4)$ will have the same determinant, since it is the conjugate transpose of $\mathbf{\Omega}(\rho_2)$. For ρ_3 we get:

$$\mathbf{\Omega}(\rho_3) = \begin{bmatrix} \omega_1 + \omega_2 + \omega_4 & \omega_1 - \omega_2 + \omega_4 \\ \omega_1 - \omega_2 + \omega_4 & \omega_1 + \omega_2 + \omega_4 \end{bmatrix},$$

which has determinant $(\omega_1+\omega_2+\omega_4)^2 - (\omega_1-\omega_2+\omega_4)^2$, which is 0 only when $\omega_1+\omega_2+\omega_4 = \pm(\omega_1 - \omega_2 + \omega_4)$. So assuming, in addition, that $\omega_2 \neq 0$, then $\omega_4 + \omega_1 = 0$.

We are looking for a tensegrity that is three-dimensional and has a positive semi-definite stress matrix, which means that it must have at least four zero eigenvalues. It always happens that $\mathbf{\Omega}(\rho_1)$ has at least one zero eigenvalue. If it has another, then $\omega_1 + \omega_2 + \omega_4 = 0$. If that happens then $\mathbf{\Omega}(\rho_2)$ will have a negative value or have $\omega_2 = 0$, neither of which is possible. So $\omega_1 + \omega_2 + \omega_4 > 0$. Thus there must be at least one other zero eigenvalue from $\mathbf{\Omega}(\rho_3)$ which implies that $\omega_4 + \omega_1 = 0$. Finally, to pick up two more zero eigenvalues, we must have $\omega_2^2 + 2\omega_2\omega_3 = (2\omega_1)^2 + \omega_2^2$. In other words, $\omega_3 = 2\omega_1^2/\omega_2$. So ω_1 and ω_2 can be chosen arbitrarily as positive numbers, while $\omega_4 = -\omega_2$, and $\omega_3 = 2\omega_1^2/\omega_2$ determine the tensegrity. In other words, there is a one-parameter family of three-dimensional super stable tensegrities.

It is also interesting to determine the configuration that corresponds to these tensegrities. Nodes $1, 2, 3, 4$ form a square when projected into the xy plane, say. This is the action of the ρ_2 and ρ_4 representations. The ρ_1 representation reflects the nodes about the xy plane. Since $\omega_4 = -\omega_1$, the four nodes with the same colour are in the same plane parallel to the xy plane. For all choices of the parameters, the $\{1, 8\}$ cable remains perpendicular to the $\{1, 3\}$ cable, while the $\{1, 8\}$ cable extends while fixing the $1, 2, 3, 4$ nodes.

11.9 Comments.

There are several interesting things to observe from the pictures in the catalogue. When the cable graph is planar, that is it has a topological embedding in the plane (or equivalently in the 2-dimensional surface of a sphere), then it appears that the cables in most of the winning representations are edges of the convex polytope determined by its vertices. For many of these examples that the winning representation is determined by the choice of the cable generators.

On the other hand, for the fifth cable class for the group $\mathcal{A}_5$, both of the two irreducible 3-dimensional representations appear, seemingly at random. But for that cable class the two generators are of order 5 and are in distinct conjugacy classes. Indeed, there is an automorphism of $\mathcal{A}_5$ that interchanges the two degree 5 generators. So which representation is the winner depends on the choice of the strut group element s_1. One can see that the order 5 elements are in distinct conjugacy classes because the corresponding cables form a convex pentagon in one case, and a self-intersecting pentagram in the other case.

The action of the automorphisms on the group $\mathcal{A}_5$ is important. When both of the generators for $\mathcal{A}_5$ are of the same order, which turns out to be both of order 3 or both of order 5, then there is an automorphism that interchanges the two generators. There are two pairs of non-C-equivalent generators of order 5 and one pair of order 3. So when we look for representatives for each of the C-equivalence classes of pairs of generators for the group $\mathcal{A}_5 \times \mathcal{S}_2$, there are three possibilities for each of the 8 C-equivalence classes of pairs of generators for $\mathcal{A}_5$, plus or minus each generator, where at least one generator for $\mathcal{A}_5 \times \mathcal{S}_2$ has a minus. But for the three cases of pairs of generators of order 3 and 5 for $\mathcal{A}_5$, when one generator receives a minus, it is C-equivalent to the case when the other generator receives a minus. This accounts for three cases that are C-equivalent. So there are $3 \cdot 8 - 3 = 21$ C-equivalence classes in all for $\mathcal{A}_5 \times \mathcal{S}_2$.

When the cables form a pentagram, which is necessarily self-intersecting, it is possible to create other tensegrities that are also stable with a positive semi-definite, maximal rank stress matrix. The idea is to replace the pentagram with a pentagon, which will evidently show up in the other C-equivalence class, or replace the pentagram with a star figure as indicated below. One is simply adding positive semi-definite quadratic forms together to get other positive semi-definite quadratic forms. See Figure 5.7. We apply that idea in Figure 11.11 and Figure 11.12. Note that it is not always possible to take a cable polygon and replace it with a star figure as in the case of a dihedral tensegrity in Zhang et al. (2010).

Figure 11.11 shows an example of a super stable highly symmetric tensegrity, where one of the cable orbits, in red, is a pentagram instead of a pentagon, as verified by a computer. Figure 11.12 shows how to add the stresses of previously known super stable tensegrities. If the stresses are scaled appropriately when cables and struts overlap, those stresses will cancel in the sum. The end result is that the red cables are replaced by star figures.

Another situation is the superstability for star figures replacing at some of the convex polygons for tensegrities with dihedral symmetry as in Connelly and Terrell (1995). In that case, for example, when the pentagonal polygon is replaced by the self-intersecting pentagram, the resulting tensegrity is definitely not superstable. So the argument above will not work to imply that, when the star replaces the pentagon, the resulting tensegrity is superstable. Nevertheless, in Zhang et al. (2010); Zhang and Ohsaki (2007, 2012), that kind of star replacement and many others are shown, and their superstability and prestress stability is determined.

For example for the tensegrity with 6-fold rotational symmetry in Figure 11.13, the top is a star and bottom is a regular hexagon, and it is superstable. If the top is replaced by a regular hexagon, it is also superstable as in Connelly and Terrell (1995). If the bottom is

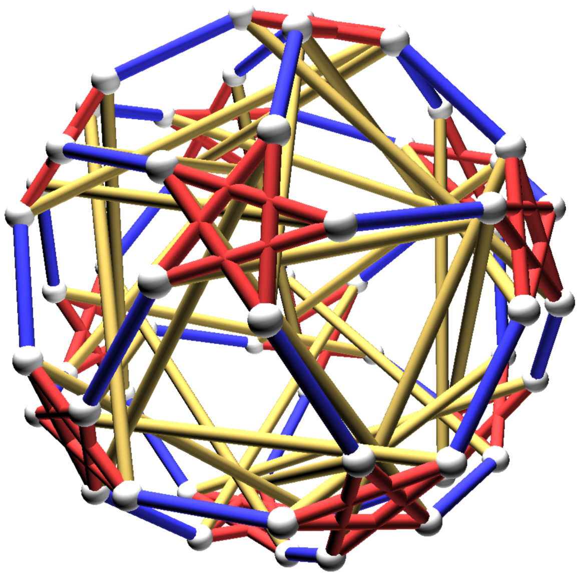


Figure 11.11: A super stable highly symmetric tensegrity in 3-space. The red and blue members are cables, and the yellow members are struts.

replaced by a star to get two stars, the resulting tensegrity is not even rigid, since the whole tensegrity breaks up into two pieces that rotate relative to each other about the line through the two star vertices. This tensegrity is what is called *divisible* by Zhang and Ohsaki.

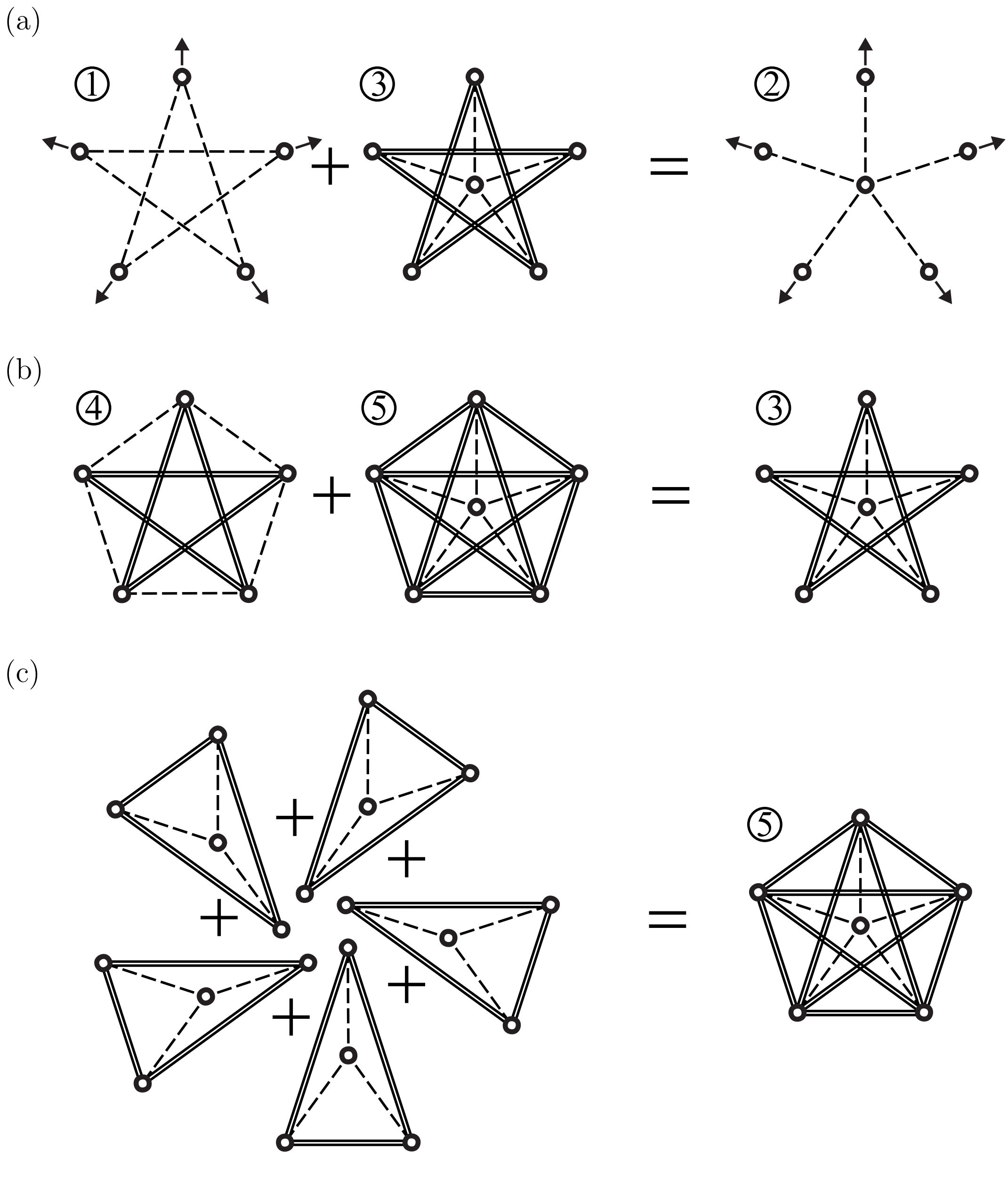


Figure 11.12: If a super stable tensegrity incorporates a pentagram of cables (such as the red cables of the tensegrity in Figure 11.11), then this figure shows that the pentagram can be replaced with a star of five cables connected to a central node without affecting the super stability. In (a), the pentagram of cables is shown as ①, and the forces that the rest of the tensegrity used to apply are shown as arrows. The tensegrity ③ is super stable. It is chosen with a symmetric stress, and then added this to ① cancelling the overlapping cables. The tensegrity ② is the result, where the five cables are connected to a central node, carrying the same forces. In (c), tensegrity ⑤ is shown to be super stable as it can be formed by the superposition of five super stable (0,2)-tensegrities (see Figure 5.8). In (b), tensegrity ③ is shown to be the sum of the super stable symmetrized Cauchy polygon ④ as in subsection 5.14.2, and the tensegrity ⑤, thus showing that tensegrity ③ is super stable.

Figure 11.13: A super stable rotationally symmetric tensegrity with a star on the top and a regular hexagon on the bototm.