Using Symmetry for Tensegrity Form-Finding

R. PANDIA RAJ[1] and S.D. GUEST[2]
Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK
[1] *Graduate Student, rp317@cam.ac.uk,* [2] *Reader, sdg@eng.cam.ac.uk*

ABSTRACT

Symmetry can simplify the form-finding process for tensegrity structures; and this paper will describe one technique. Our method is based on the commonly used force density method, but the calculations are done using a *symmetry-adapted coordinate system*. The standard force-density method assumes a known connectivity for the structure. A tension coefficient (tension divided by length) must then be found for every member so that an equilibrium solution is possible. Finding the nodal coordinates is straightforward once a suitable set of tension coefficient is found: but finding suitable tension coefficients may be non-trivial. In this paper we simplify the correct choice of tension coefficients by the use of symmetry — in addition to the connectivity of the structure, we assume that the structure has certain symmetry properties, greatly reducing the difficulty of finding possible configurations. The paper will show simple examples of the method where a simple analytical solution gives all possible symmetric tensegrities with a given connectivity.

1. INTRODUCTION

Tensegrity structures are rigidized by self-stress. The key step in design of these structures is form-finding, the determination of a self-stressed equilibrium configuration. Here, a simple technique for tensegrity form-finding is described based on the commonly used force density method, but the calculations are done using a *symmetry-adapted coordinate system*.

The standard force-density method, presented in [1], assumes a known connectivity for the structure. A tension coefficient (tension divided by length) must then be found for every member so that an equilibrium solution is possible. Finding the nodal coordinates is straightforward once a suitable set of tension coefficient is found: but finding suitable tension coefficients may be non-trivial. We will show that using symmetry can help.

2. EQUILIBRIUM AND STABILITY OF TENSEGRITY STRUCTURES

In the force-density method, the equilibrium of the structure is written using a *stress matrix*, $\mathbf{S}$ (often known as the force density matrix), which is defined as follows. Consider two nodes i and j, possibly connected by a member ij which carries a tension coefficient $\hat{t}_{ij}$. The coefficients of the stress matrix are

$$\mathbf{S}_{ij} = \begin{cases} -\hat{t}_{ij} & \text{if } i \neq j, \\ \sum \hat{t}_{ik} & \text{if } i = j : \text{summation over all nodes } k \text{ connected to node } i \\ 0 & \text{if } i \text{ and } j \text{ are not connected} \end{cases} \tag{1}$$

If the unknown nodal coordinates are written as three vectors, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, and applied nodal forces in the x, y and z directions are written as $\mathbf{p_x}$, $\mathbf{p_y}$, and $\mathbf{p_z}$, then unloaded equilibrium configurations are solutions of the equations.

$$\begin{aligned} \mathbf{Sx} &= \mathbf{p_x} = \mathbf{0} \\ \mathbf{Sy} &= \mathbf{p_y} = \mathbf{0} \\ \mathbf{Sz} &= \mathbf{p_z} = \mathbf{0} \end{aligned} \tag{2}$$

In order for a three-dimensional structure to exist, the equilibrium equations (Eq. 2) must have three independent solutions, which themselves must be independent of the uniform vector $[1, 1, ...1]^{\mathbf{T}}$, which will always be in the nullspace of any properly defined stress matrix. Thus the tension coefficients in the members must be chosen such that $\mathbf{S}$ has a nullity of 4, i.e. it is rank-deficient by 4. In addition, to guarantee that the structure is stable, we require $\mathbf{S}$ to be positive semi-definite.

In this paper we simplify the correct choice of tension coefficients by the use of symmetry. In addition to the connectivity of the structure, we assume that the structure has certain symmetry properties. We then

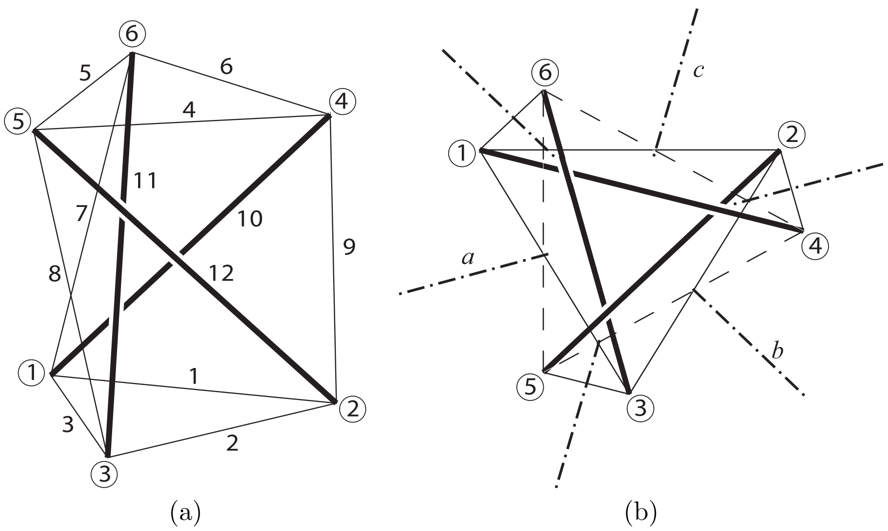

Figure 1: Example tensegrity structure with D_3 symmetry. (a) Isometric view, showing the node and element numbering scheme used. Tension members are shown by thin lines, compression members by thick lines. (b) Plan view, showing the top triangle by dotted lines, bottom triangle by solid lines, and the location of three C_2 rotation axes, a, b, c, each of which lies at half height in the structure.

write the stress matrix, $\mathbf{S}$ using a *symmetry-adapted coordinate system* that is defined by the *irreducible representations* of the symmetry group to which the structure belongs: the resultant stress matrix $\tilde{\mathbf{S}}$ is *similar* to $\mathbf{S}$, but has a block-diagonal form. The nullity of the whole matrix is now simply the sum of nullity of each the sub-blocks of $\tilde{\mathbf{S}}$, and hence finding the required nullity of 4 is simplified.

3. SIMPLEX TENSEGRITY

Consider the example tensegrity shown in Fig. 1: we will assume that it has a totally symmetric state of self-stress where the tension coefficients are given by $\hat{T}_d$ for the diagonal struts (members 10, 11, and 12), $\hat{T}_h$ for the horizontal cables (members 1–6), and $\hat{T}_v$ for the vertical cables (members 7, 8, and 9). The stress matrix (defined in Eq. 1) is then

$$\mathbf{S} = \begin{bmatrix} 2\hat{T}_h+\hat{T}_v+\hat{T}_d & -\hat{T}_h & -\hat{T}_h & -\hat{T}_d & 0 & -\hat{T}_v \\ -\hat{T}_h & 2\hat{T}_h+\hat{T}_v+\hat{T}_d & -\hat{T}_h & -\hat{T}_v & -\hat{T}_d & 0 \\ -\hat{T}_h & -\hat{T}_h & 2\hat{T}_h+\hat{T}_v+\hat{T}_d & 0 & -\hat{T}_v & -\hat{T}_d \\ -\hat{T}_d & -\hat{T}_v & 0 & 2\hat{T}_h+\hat{T}_v+\hat{T}_d & -\hat{T}_h & -\hat{T}_h \\ 0 & -\hat{T}_d & -\hat{T}_v & -\hat{T}_h & 2\hat{T}_h+\hat{T}_v+\hat{T}_d & -\hat{T}_h \\ -\hat{T}_v & 0 & -\hat{T}_d & -\hat{T}_h & -\hat{T}_h & 2\hat{T}_h+\hat{T}_v+\hat{T}_d \end{bmatrix} \quad (3)$$

Note that the stress matrix does not depend on the actual configuration of the structure.

In the *Schoenflies notation* [2] the structure has D_3 symmetry—it is transformed into an *equivalent configuration* by six symmetry operations: the identity, E, rotation by 120° (C_3) or rotation by 240° (C_3^2) about the vertical axis; twofold rotation about the three axes a (C_2^a), b (C_2^b), and c (C_2^c). The *irreducible representations* of a symmetry group [3] shown in Table 1, provide the means to find a symmetry adapted coordinate system, as described in [4]. Applying that method here gives an orthogonal transformation matrix $\mathbf{V}$, so that $\tilde{\mathbf{x}} = \mathbf{V^T x}$, $\tilde{\mathbf{p}}_\mathbf{x} = \mathbf{V^T p_x}$, where

$$\mathbf{V} = 1/\sqrt{6} \begin{array}{c} \begin{array}{cccc} \mathrm{A}_1 & \mathrm{A}_2 & \mathrm{E}(1) & \mathrm{E}(2) \end{array} \\ \left[\begin{array}{c|c|cc|cc} 1 & 1 & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & \sqrt{3/2} & -1/\sqrt{2} & -1/\sqrt{2} & -\sqrt{3/2} \\ 1 & 1 & -\sqrt{3/2} & -1/\sqrt{2} & -1/\sqrt{2} & \sqrt{3/2} \\ 1 & -1 & 0 & \sqrt{2} & -\sqrt{2} & 0 \\ 1 & -1 & -\sqrt{3/2} & -1/\sqrt{2} & 1/\sqrt{2} & -\sqrt{3/2} \\ 1 & -1 & \sqrt{3/2} & -1/\sqrt{2} & 1/\sqrt{2} & \sqrt{3/2} \end{array}\right] \end{array} \quad (4)$$

$\mathbf{V}$ defines four symmetry subspaces. A_1 is where loads, or coordinates, are totally symmetric—unchanged by every symmetry operation; for A_2, loads and coordinates are preserved by E, C_3 and C_3^2, but reversed

Table 1: Irreducible representations of symmetry group D_3

D_3	E	C_3	C_3^2	C_2^a	C_2^b	C_2^c
A_1	1	1	1	1	1	1
A_2	1	1	1	-1	-1	-1
E	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$	$\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$	$\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

by C_2^a, C_2^b and C_2^c. E is a two-dimensional representation, whose symmetry subspace gathers anything not in A_1 or A_2: it splits into E(1), quantities preserved by C_2^a, and E(2), quantities reversed by C_2^a.

The symmetry adapted $\tilde{\mathbf{S}}$ where $\tilde{\mathbf{S}}\tilde{\mathbf{x}} = \tilde{\mathbf{p}}_{\mathbf{x}}$ etc., can be written as

$$\tilde{\mathbf{S}} = \mathbf{V^T S V} \tag{5}$$

which gives,

$$\tilde{\mathbf{S}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\hat{T}_v + 2\hat{T}_d & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\hat{T}_h + \hat{T}_v/2 + 2\hat{T}_d & -(\sqrt{3}/2)\hat{T}_v & 0 & 0 \\ 0 & 0 & -(\sqrt{3}/2)\hat{T}_v & 3\hat{T}_h + \hat{T}_v/2 + 2\hat{T}_d & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\hat{T}_h + \hat{T}_v/2 + 2\hat{T}_d & -(\sqrt{3}/2)\hat{T}_v \\ 0 & 0 & 0 & 0 & -(\sqrt{3}/2)\hat{T}_v & 3\hat{T}_h + \hat{T}_v/2 + 2\hat{T}_d \end{bmatrix} \tag{6}$$

Eq. 6 shows that the block-diagonalized stress matrix, $\tilde{\mathbf{S}}$ consists of four independent sub-matrix blocks, the $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{E}(1)$, and $\mathbf{E}(2)$ blocks — note that $\mathbf{E}(1)$ and $\mathbf{E}(2)$ are identical, which is a consequence of the symmetry. We will now consider each of these blocks separately.

$\mathbf{A}_1$ block

$$\tilde{\mathbf{S}}^{A_1} = [0] \tag{7}$$

The value of first (1×1) matrix, $\tilde{\mathbf{S}}^{A_1}$ must be zero (nullity = 1) for a properly configured stress matrix, because the sum of any row (or column) is zero, by definition.

$\mathbf{A}_2$ block

$$\tilde{\mathbf{S}}^{A_2} = [2\hat{T}_v + 2\hat{T}_d] \tag{8}$$

When

$$\hat{T}_v = -\hat{T}_d, \tag{9}$$

the second block gives nullity 1.

E blocks

$$\tilde{\mathbf{S}}^{E_1} = \tilde{\mathbf{S}}^{E_2} = \begin{bmatrix} 3\hat{T}_h + \hat{T}_v/2 + 2\hat{T}_d & -(\sqrt{3}/2)\hat{T}_v \\ -(\sqrt{3}/2)\hat{T}_v & 3\hat{T}_h + \hat{T}_v/2 + 2\hat{T}_d \end{bmatrix} \tag{10}$$

In order to have a nullity of greater than zero for each $\mathbf{E}(1)$ and $\mathbf{E}(2)$ blocks, the determinant of $\tilde{\mathbf{S}}^{E_1}$ and $\tilde{\mathbf{S}}^{E_2}$ should be zero.

$$\begin{vmatrix} 3\hat{T}_h + \hat{T}_v/2 + 2\hat{T}_d & -(\sqrt{3}/2)\hat{T}_v \\ -(\sqrt{3}/2)\hat{T}_v & 3\hat{T}_h + \hat{T}_v/2 + 2\hat{T}_d \end{vmatrix} = 0$$

$$\left(3\hat{T}_h + \hat{T}_v/2 + 2\hat{T}_d\right)^2 - \left(-\sqrt{3}/2\hat{T}_v\right)^2 = 0 \tag{11}$$

To obtain a total nullity of 4, we need to satisfy equations (9) and (11), with non-zero tension coefficients. The solutions are

$$\begin{aligned} \hat{T}_v/\hat{T}_d &= -1 \\ \hat{T}_v/\hat{T}_h &= \pm\sqrt{3} \end{aligned} \tag{12}$$

These solutions result in $\mathbf{S}$ having a nullity of 4, and also being positive definite. There is choice in the value of $\hat{T}_v/\hat{T}_h$; however, if we additionally require both horizontal cables and vertical cables to be in tension, with $\hat{T}_v$ and $\hat{T}_h$ positive, then we must choose $\hat{T}_v/\hat{T}_h = +\sqrt{3}$. ($\hat{T}_v/\hat{T}_h = -\sqrt{3}$ is also a valid structural configuration, but with the role of diagonal struts and vertical cables reversed.)

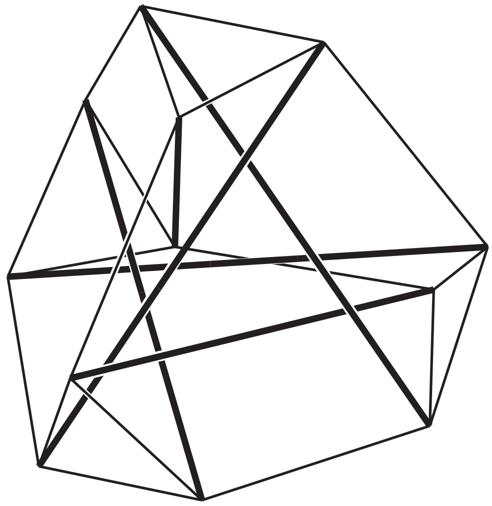

Figure 2: Tensegrity structure with T symmetry. The outer net of members shown by thin lines are cables in tension and the inner bars shown by thick lines are struts in compression.

$\hat{T}_v/\hat{T}_d = -1$ and $\hat{T}_v/\hat{T}_h = +\sqrt{3}$ determine the stress matrix for the structure (Eq. 3) apart from a single parameter representing the overall magnitude of the state of self-stress. This parameter does not affect the nullspace of $\mathbf{S}$, and hence it is possible to find stable equilibrium configurations of the system from the possible solutions of Eq. 2.

4. T GROUP TENSEGRITY STRUCTURES

As a second example, the structure shown in Fig. 2 will be analysed. It is a structure with point group symmetry T, the symmetries of rotations, but not reflections, of a tetrahedron. Let the tension coefficients due to the prestress be denoted by $\hat{T}_t$, $\hat{T}_d$ and, $\hat{T}_s$, for the cables in the triangles, the cables connecting the triangles, and the struts, respectively. The stress matrix, $\mathbf{S}$ can be set up in terms of only these three tension coefficients.

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \tag{13}$$

where,

$$\mathbf{S}_{11} = \begin{bmatrix} 2\hat{T}_t+\hat{T}_d+\hat{T}_s & -\hat{T}_t & -\hat{T}_t & 0 & -\hat{T}_d & 0 \\ -\hat{T}_t & 2\hat{T}_t+\hat{T}_d+\hat{T}_s & -\hat{T}_t & 0 & 0 & -\hat{T}_d \\ -\hat{T}_t & -\hat{T}_t & 2\hat{T}_t+\hat{T}_d+\hat{T}_s & -\hat{T}_d & 0 & 0 \\ 0 & 0 & -\hat{T}_d & 2\hat{T}_t+\hat{T}_d+\hat{T}_s & 0 & 0 \\ -\hat{T}_d & 0 & 0 & 0 & 2\hat{T}_t+\hat{T}_d+\hat{T}_s & 0 \\ 0 & -\hat{T}_d & 0 & 0 & 0 & 2\hat{T}_t+\hat{T}_d+\hat{T}_s \end{bmatrix},$$

$$\mathbf{S}_{12} = \begin{bmatrix} -\hat{T}_s & 0 & 0 & 0 & 0 & 0 \\ 0 & -\hat{T}_s & 0 & 0 & 0 & 0 \\ 0 & 0 & -\hat{T}_s & 0 & 0 & 0 \\ 0 & -\hat{T}_t & 0 & -\hat{T}_t & 0 & -\hat{T}_s \\ 0 & 0 & -\hat{T}_t & -\hat{T}_s & -\hat{T}_t & 0 \\ -\hat{T}_t & 0 & 0 & 0 & -\hat{T}_s & -\hat{T}_t \end{bmatrix}, \quad \mathbf{S}_{21} = \begin{bmatrix} -\hat{T}_s & 0 & 0 & 0 & 0 & -\hat{T}_t \\ 0 & -\hat{T}_s & 0 & -\hat{T}_t & 0 & 0 \\ 0 & 0 & -\hat{T}_s & 0 & -\hat{T}_t & 0 \\ 0 & 0 & 0 & -\hat{T}_t & -\hat{T}_s & 0 \\ 0 & 0 & 0 & 0 & -\hat{T}_t & -\hat{T}_s \\ 0 & 0 & 0 & -\hat{T}_s & 0 & -\hat{T}_t \end{bmatrix} \text{ and}$$

$$\mathbf{S}_{22} = \begin{bmatrix} 2\hat{T}_t + \hat{T}_d + \hat{T}_s & 0 & 0 & -\hat{T}_d & 0 & -\hat{T}_t \\ 0 & 2\hat{T}_t + \hat{T}_d + \hat{T}_s & 0 & -\hat{T}_t & -\hat{T}_d & 0 \\ 0 & 0 & 2\hat{T}_t + \hat{T}_d + \hat{T}_s & 0 & -\hat{T}_t & -\hat{T}_d \\ -\hat{T}_d & -\hat{T}_t & 0 & 2\hat{T}_t + \hat{T}_d + \hat{T}_s & 0 & 0 \\ 0 & -\hat{T}_d & -\hat{T}_t & 0 & 2\hat{T}_t + \hat{T}_d + \hat{T}_s & 0 \\ -\hat{T}_t & 0 & -\hat{T}_d & 0 & 0 & 2\hat{T}_t + \hat{T}_d + \hat{T}_s \end{bmatrix}$$

The symmetry group T has three irreducible representations, the one-dimensional A, the two-dimensional E, and the three-dimensional T. It is straightforward to find a symmetry transformation matrix to find the block-diagonalized stress matrix $\tilde{\mathbf{S}}$, which has the structure

$$\tilde{\mathbf{S}} = \begin{bmatrix} \tilde{\mathbf{S}}^{\mathrm{A}} & & & & & \\ & \tilde{\mathbf{S}}^{\mathrm{E(1)}} & & & & \\ & & \tilde{\mathbf{S}}^{\mathrm{E(2)}} & & & \\ & & & \tilde{\mathbf{S}}^{\mathrm{T(1)}} & & \\ & & & & \tilde{\mathbf{S}}^{\mathrm{T(2)}} & \\ & & & & & \tilde{\mathbf{S}}^{\mathrm{T(3)}} \end{bmatrix} \tag{14}$$

$\tilde{\mathbf{S}}$ is a (12×12) matrix; $\tilde{\mathbf{S}}^{\mathrm{A}}$ is (1×1); $\tilde{\mathbf{S}}^{\mathrm{E(1)}}$ and $\tilde{\mathbf{S}}^{\mathrm{E(2)}}$ are (2×2); $\tilde{\mathbf{S}}^{\mathrm{T(1)}}$, $\tilde{\mathbf{S}}^{\mathrm{T(2)}}$, and $\tilde{\mathbf{S}}^{\mathrm{T(3)}}$ are (3×3). $\tilde{\mathbf{S}}^{\mathrm{A}}$ is guaranteed to have a nullity of 1 for any properly constructed stress matrix. $\tilde{\mathbf{S}}^{\mathrm{T(1)}}$, $\tilde{\mathbf{S}}^{\mathrm{T(2)}}$, and $\tilde{\mathbf{S}}^{\mathrm{T(3)}}$ are guaranteed to be *similar* to one another, and hence if the nullity of $\tilde{\mathbf{S}}^{\mathrm{T(1)}} = 1$, then the total nullity of $\mathbf{S}$ will be at least the required 4. Thus, we require

$$\left|\tilde{\mathbf{S}}^{\mathrm{T(1)}}\right| = 0 \tag{15}$$

For convenience, $\tilde{\mathbf{S}}^{\mathrm{T(1)}}$ can be written as

$$\tilde{\mathbf{S}}^{\mathrm{T(1)}} = \hat{T}_t\mathbf{A} + \hat{T}_d\mathbf{B} + \hat{T}_s\mathbf{C} \tag{16}$$

and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are found to be

$$\mathbf{A} = \begin{bmatrix} 3.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0372 & -0.3319 \\ 0.0000 & -0.3319 & 2.9628 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1.0000 & 0.8757 & -0.4829 \\ 0.8757 & 1.2332 & 0.4229 \\ -0.4829 & 0.4229 & 1.7668 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 2.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.5337 & -0.8457 \\ 0.0000 & -0.8457 & 0.4663 \end{bmatrix}$$

The overall magnitude of stress is not important, so we can write Eq. 15 as

$$\left|\mathbf{A} + (\hat{T}_d/\hat{T}_t)\mathbf{B} + (\hat{T}_s/\hat{T}_t)\mathbf{C}\right| = 0 \tag{17}$$

The solutions to Eq. 17 are plotted in Fig. 3. Requiring $\mathbf{S}$ to be positive semi-definite, with cables carrying tension and struts compression gives a single line of possible solutions. Assuming suitable values on this line, say $\hat{T}_d/\hat{T}_t = 2.0$ and $\hat{T}_s/\hat{T}_t = -0.759$, gives a positive semi-definite stress matrix with the required nullity 4. The nullspace of this stress matrix dictates possible coordinates for this structure, of which one set gives the tensegrity structure shown in Fig. 2. This is not the only possible configuration, but any other configuration must be a stretched and rotated version of the structure shown.

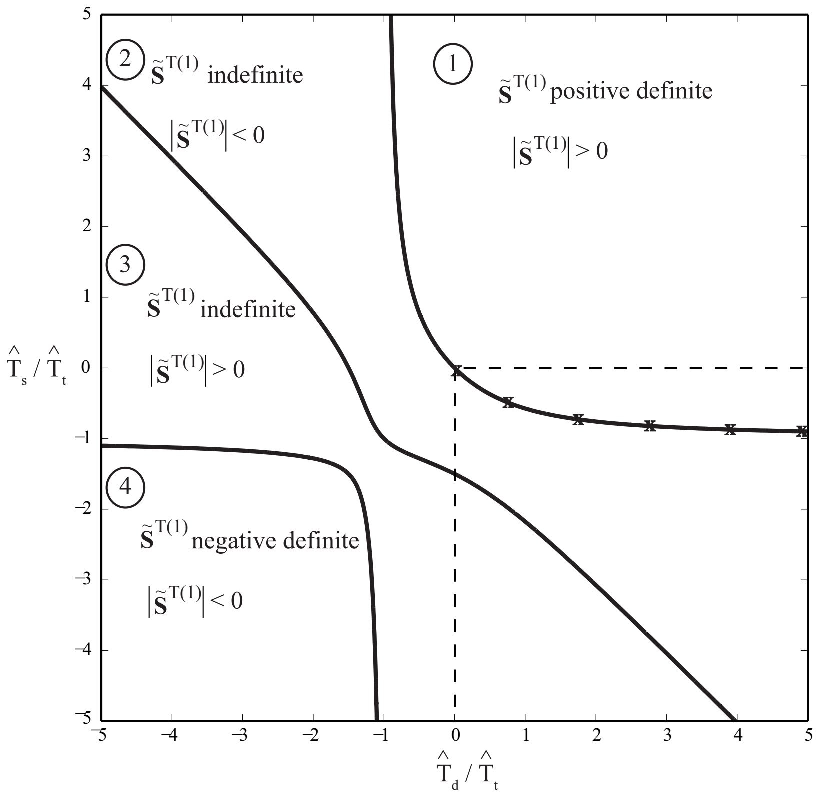

Figure 3: Solutions of Eq. 17, $|\tilde{\mathbf{S}}^{T(1)}| = 0$. The solutions lie on three lines, which split the plot into 4 regions. Requiring $\mathbf{S}$, and hence $\tilde{\mathbf{S}}^{T(1)}$ to be positive semi-definite implies that we are interested in the solution between regions (1) and (2). Additionally requiring $\hat{T}_d/\hat{T}_t$ positive, and $\hat{T}_s/\hat{T}_t$ negative, gives solutions marked by crosses.

5. CONCLUSION

In this paper, it is shown using the stress matrix that for a three-dimensional structure to exist, the stress matrix must have nullity of four. The nullspace of the stress matrix then gives the configuration of the tensegrity. Using a symmetry adapted coordinate system makes it easy to check the right nullity. Furthermore, it also helps to find a set of tension coefficients that achieve equilibrium configurations of tensegrity structures which are prestress stable and hence greatly reduces the difficulty of finding possible configurations.

6. ACKNOWLEDGEMENTS

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