INEXTENSIONAL WRAPPING OF FLAT MEMBRANES

S.D. Guest* and S. Pellegrino†

*Department of Engineering, University of Cambridge,
Trumpington Street, Cambridge, CB2 1PZ, U.K.

†Structures and Mechanisms Division, ESA-ESTEC,
P.O. Box 299, 2200 AG Noordwijk, The Netherlands.
Permanent address: Department of Engineering, University of Cambridge,
Trumpington Street, Cambridge, CB2 1PZ, U.K.

ABSTRACT

In this paper we are concerned with the wrapping of a flat, thin membrane around a central hub. The folding pattern consists of a series of hill and valley folds, as in a recent proposal by Temple and Oswald for the design of a solar sail. For launch their sail is wrapped around the circular body of the spacecraft, about 4 m in diameter; once in orbit it is deployed to a 276 m diameter disk which can collect enough solar pressure to sail to Mars.

The paper begins with a brief survey of related work: it turns out that, since the early 1960's, several people have thought about folding thin sheets in this way, and yet no complete solution or even explanation is available. We present a simple description of the folding technique. Based on the simplifying assumption that the membrane to be folded has zero thickness, we identify some key properties of the folding pattern and hence show how to draw the fold pattern. Then we present a simple way of computing the correct fold pattern for thin membranes. We discuss some alternative fold patterns, including irregular hubs and other variants.

In the Appendix we give two fold patterns from which simple demonstration models can be made.

From: R. Motro & T. Wester, Structural Morphology / Morphologie Structurale, Proceedings of the First International Seminar on Structural Morphology, Montpellier, 7–11 September 1992, pp 203–215

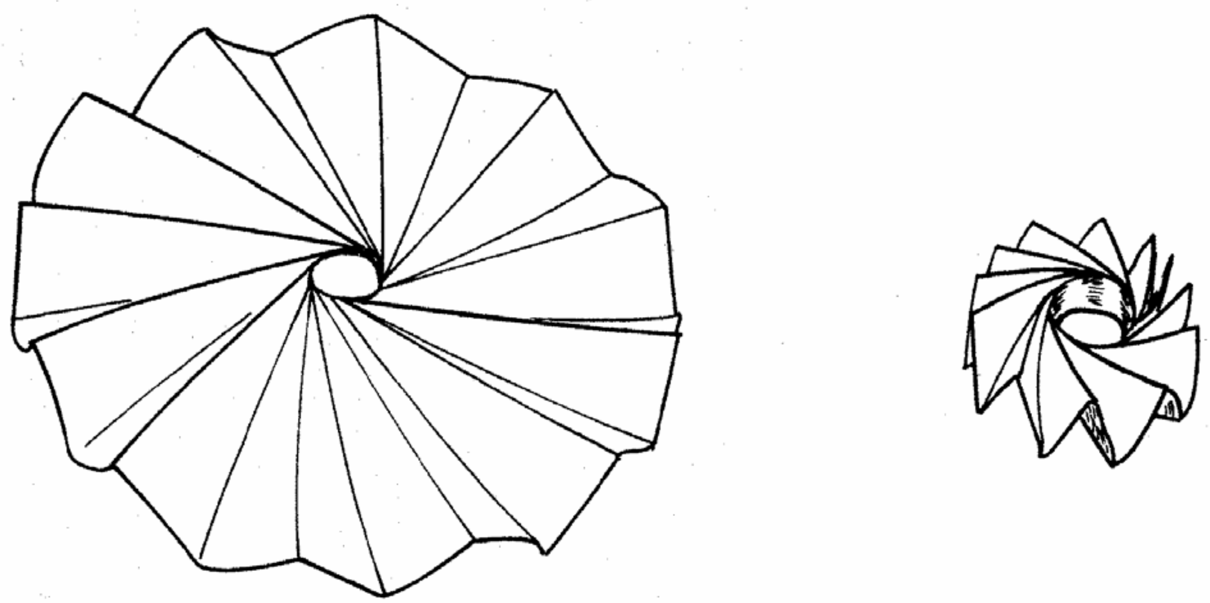


Fig.1. The basic fold pattern, for n = 24.

1. INTRODUCTION

The folding of thin, flat sheets is a matter of daily routine: a letter is often folded about its centre line and then again about the new centre line to fit a standard size envelope; a road map is usually folded into a narrow concertina pattern, with fold lines parallel to the shorter sheet side, and then the other way about a fold line perpendicular to the previous set.

In this paper we are concerned with the wrapping of a flat, thin membrane around a central hub as shown in Fig. 1. It can be seen that the folding pattern consists of a symmetric set of hill and valley folds. This folding pattern was proposed by Temple and Oswald (Cambridge Consultants, 1989) for the design of a solar sail. For launch the sail is wrapped around the circular body of the spacecraft, about 4 m in diameter; once in orbit it deploys into a 276 m diameter disk which can collect enough solar pressure to sail to Mars. See Wright (1991) for a general introduction to solar sailing.

Since the early 1960's several people have thought about folding thin sheets in this way. The idea can be found, very much in an embryonic form, in Huso (1960). Huso had invented a sheet reel for folding compactly the tarpaulin cover of a car. His device consists of a fixed part, connected to the car roof, and a rotatable hub connected to the tarpaulin: when the hub is rotated the tarpaulin is gradually wound onto it. A series of prongs have the function of guiding the tarpaulin while it is drawn towards the hub, to achieve a fairly uniform folding. Figure 4 of Huso (1960) is a sketch of the expected folding pattern: each fold appears to start tangential to the hub.

The technique was refined by Lanford (1961) who patented the folding apparatus shown in Fig. 2, where the regular spacing of hill and valley folds is achieved by means of guiding wires tensioned by weights. This produces a fully-folded sheet with a regular saw tooth edge. The fold lines, sketched in Fig. 2(a) are not straight, and were described as of helical shape by Lanford.

A third contribution was made by Scheel (1974), whose patent "Space-saving storage of flexible sheets" envisages a set of straight flexible ribs and major folds approximately tangent to the hub; a further set of intermediate folds along which the sheet is folded in alternate directions, intermediate folds bisect the angles between adjacent major folds; and, possibly, some additional minor folds, parallel to the major folds, see Fig. 3. This pattern leads to a series of pleats and sub-pleats, which are wrapped around a circular hub, as in the previous two patents. It will be shown in Section 2.2. that Scheel's major and

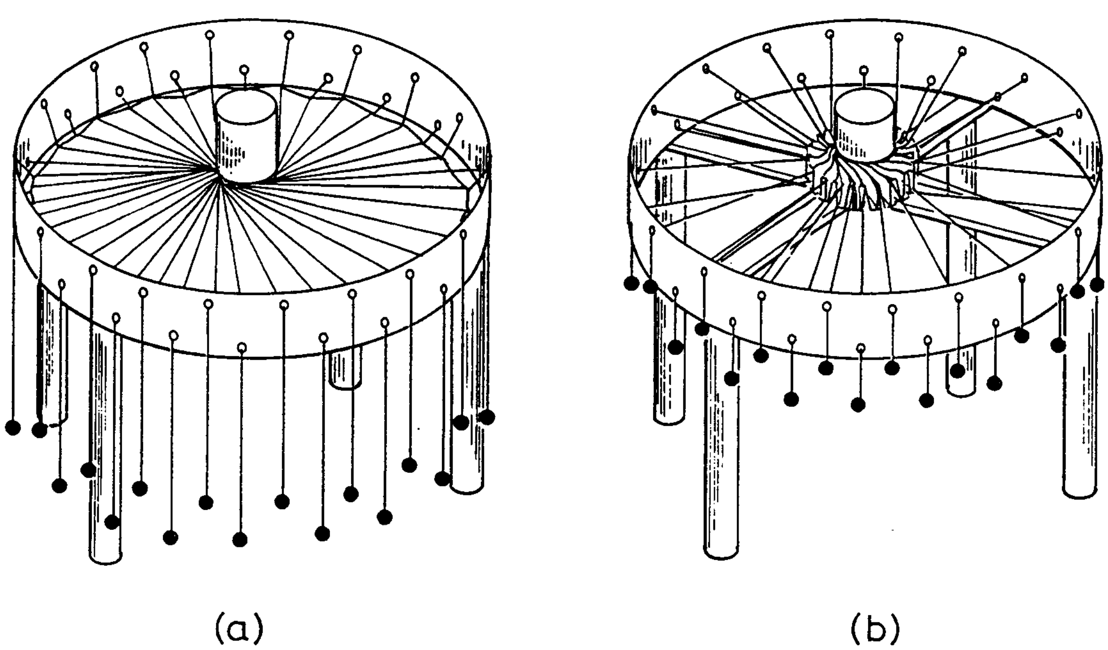


Fig. 2. Lanford's folding apparatus.

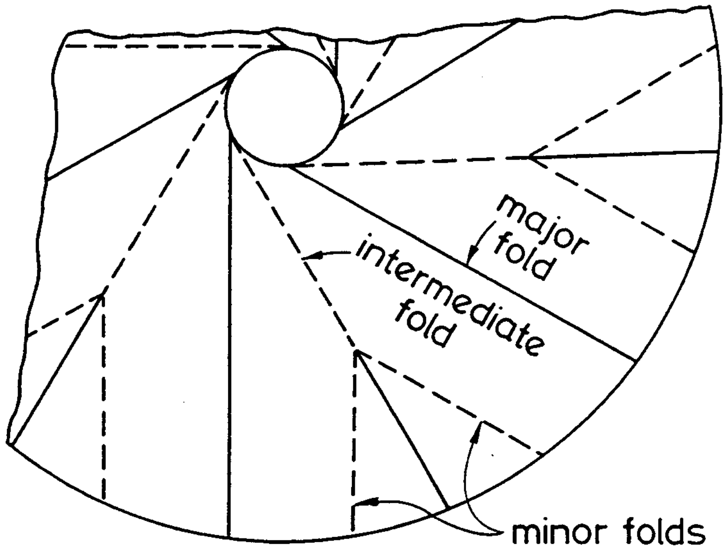


Fig. 3. Scheel's fold pattern. Valley folds shown by dashed lines.

intermediate folds can be obtained as a degenerate case of a standard fold pattern based on Lanford's folding.

Later on, a mesh antenna which is folded by wrapping its ribs around a hub was developed (Wade and McKean, 1981). No further progress was made until Temple and Oswald put forward their solar sail proposal. The key difference in their approach is that, instead of relying on folding devices, they derived an approximate equation for a fold line and, having made a template, were able to prefold a membrane according to the correct pattern. They made a series of simple paper models which could be folded and unfolded very easily by hand, without any prongs or guiding wires.

Our interest in this problem began after a lecture during which Temple demonstrated one of his models. We decided to look for a simple description of the folding technique and for ways of computing the correct fold pattern. We believe that we have been successful on both fronts. We have found that, with the simplifying assumption that the membrane to be folded has zero thickness, some key properties of the folding pattern can be identified and hence the fold lines can be drawn quite easily. In this context we have also explored some alternative, non-symmetric fold patterns. All of these matters are discussed in Section 2 of this paper. The assumption of a zero-thickness membrane is, actually, not acceptable and in fact simple models produced according to Section 2 do not work well in practice, even when made from

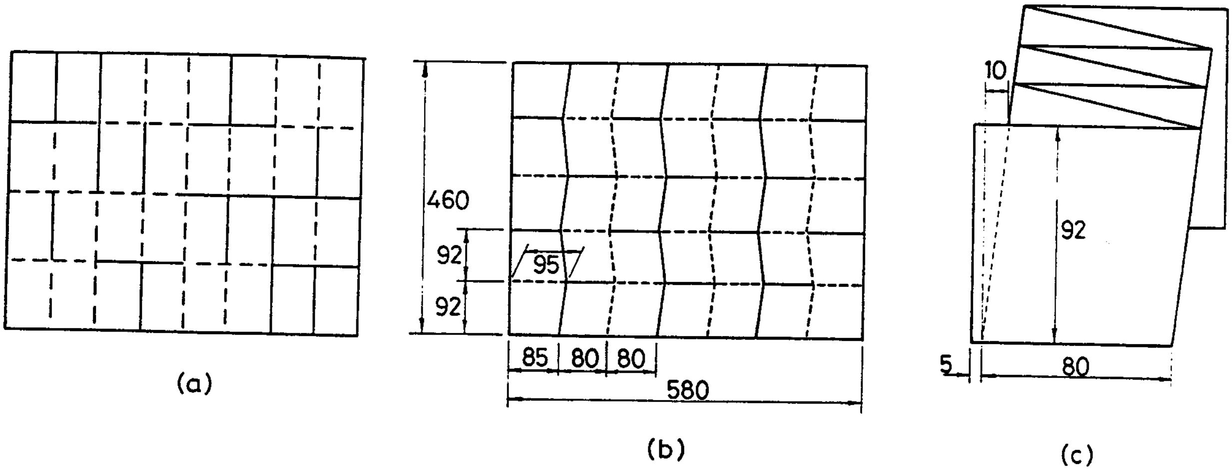


Fig. 4. (a) Standard, two-way fold pattern. (b) Fold pattern from Miura et al. (1980), fully extended. (c) Same pattern, fully folded. Valley folds shown by dashed lines.

standard photocopying paper sheets, whose thickness is approximately 0.1 mm. Hence, in Section 3 we discuss a rather simple way of accounting for membrane thickness: although conceptually unchanged, our approach is now a little more formal and, following some simple computations, the position of each fold line is found and drawn by plotter. A Discussion of the approach adopted and some suggestions for further work conclude the paper.

Before starting our detailed investigation into the wrapping of membranes it is useful to discuss a rather different folding technique (Miura, 1980) because the necessary conditions for inextensional folding apply to our problem as well. Miura noticed that the standard way of packaging a thin rectangular sheet, by folding it about its centre line and then repeating this process, say, for a total of five times, produces the fold pattern shown in Fig. 4(a). The fold lines intersect at twenty-one inner vertices: at each vertex there are two pairs of collinear folds. Figure 4(b) shows the slightly more elaborate and yet much better fold pattern devised by Miura, which produces a coupled biaxial contraction of the sheet and requires hinges of uniform thickness, unlike the pattern of Fig. 4(a) where the folds formed last have to bend round several layers of previously formed folds.

Miura (1989) shows that, for inextensional folding, three of the four fold lines intersecting at a vertex must be of one sign (e.g. concave, or valley folds) and the fourth of opposite sign (convex, or hill fold). With a single exception, in Section 2.2, all fold patterns considered in this paper are precisely of this type. The same paper shows that if there are only two fold lines meeting at a vertex, they must be collinear; that there can be no vertices with three fold lines, unless — of course— two of them are collinear and the third is not active; and finally that there can be no vertex where all fold lines have equal sign, unless there are two folds only. Miura's results, originally obtained for the discrete case of straight folds intersecting at vertex points, can be extended by a limit process to discuss the properties of curved fold lines as well, see Section 4.

2. MEMBRANES OF ZERO THICKNESS

Because we are interested, obviously, in fold patterns with a finite number of folds and also in preserving the flatness of the hub, folds which are continuously curved around the hub are not possible (Johnson and Yu, 1980). Therefore, it is necessary to assume that at the centre of the fold pattern there will be an n-sided polygon with straight sides. Figure 5(a) shows the fold pattern for a membrane of zero thickness, to be wrapped around a hexagonal hub, in this section we show how this pattern was obtained. In analogy with Scheel (1974), we shall call *major folds* the $n = 6$ folds which originate at hub vertices and extend to the edge of the fold pattern. Note that, when the membrane is absolutely flat and hence fold signs do not matter, the folding pattern itself has n-fold rotational symmetry.

The treatment presented in this section is applicable to any polygonal hub, the only requirement is that the number of sides n, and hence the number of major folds, should be even because each hill fold needs a corresponding valley fold. Initially, we will look at regular polygons; irregular polygons will be considered in Section 2.1.

We begin by calculating the angles between fold lines intersecting at a hub vertex. Referring to vertex A, Fig. 5(b), given the hub angle α, we wish to calculate the angles β, χ, δ. Obviously,

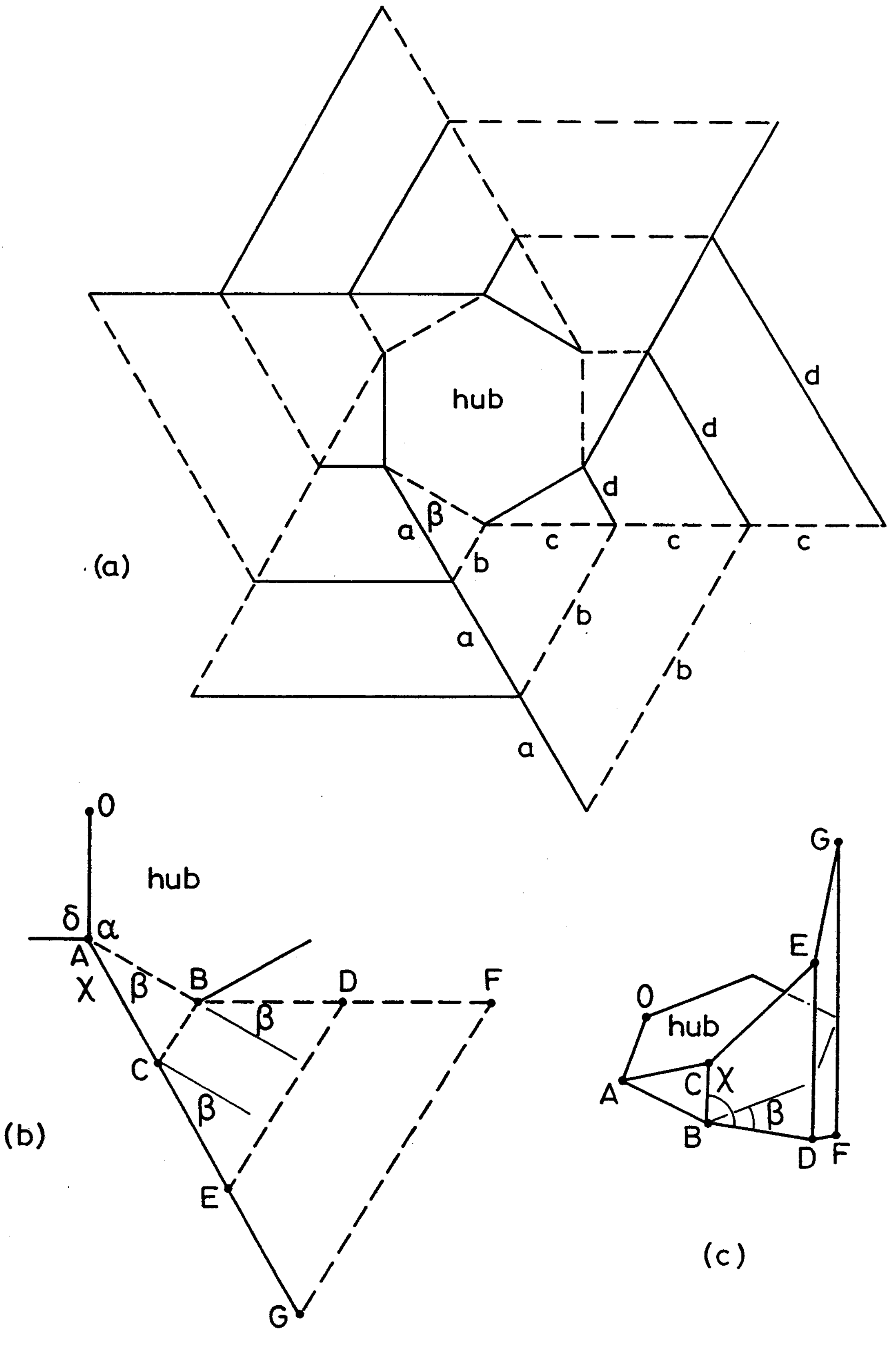


Fig. 5.(a) Fold pattern for n = 6 and t = 0. (b) Detail of (a). (c) Wrapped configuration of membrane panels in (b). Valley folds shown by dashed lines.

$$\alpha = \left(1 - \frac{2}{n}\right)\pi \tag{1}$$

because the sum of the inner angles in an n-sided, regular polygon is $(n - 2)\pi$. Also

$$\alpha + \beta + \chi + \delta = 2\pi \tag{2}$$

because the angular defect at every point of the membrane, point A in particular, is zero (Calladine, 1983).

To obtain two more equations we consider the fully wrapped configuration. Because the membrane has zero thickness, it coincides —in plan view— with the edge of the hub. Hence BC, DE, FG end up vertical, which implies

$$\angle ABC = \delta = \frac{\pi}{2}. \tag{3}$$

Because, by symmetry, the angles at vertex A are equal to the corresponding angles at B, Fig. 5(b), and BC is vertical after wrapping, we have the fourth condition, see Fig. 5(c):

$$\chi - \beta = \frac{\pi}{2}. \tag{4}$$

Given equation (1), the solution of the system of equations (2-4) is

$$\beta = \frac{\pi}{n}, \quad \chi = \left(\frac{1}{2} + \frac{1}{n}\right)\pi, \quad \delta = \frac{\pi}{2} \tag{5}$$

which defines completely the fold pattern in the region next to the hub. Note that AC bisects the angle between side AB and the line of side OA.

Next, to define the remainder of the folding pattern, we note that the folds BC, DE, FG end up parallel in Fig. 5(c) and hence have to be parallel, since they are coplanar, also in Fig. 5(b). They are also equidistant because in Fig. 5(c) they pass through adjacent vertices of the hub. With reference to Fig. 5(a), this shows that type b folds are parallel and equidistant. By symmetry, the same is true for type d folds.

Finally, we note that B, D, F and, similarly, A, C, E, G, are collinear because type c folds pass through the intersections of folds b and d, in Fig. 5(a).

Now we are ready to draw a complete folding pattern on a flat sheet. First we draw an n-sided regular polygon representing the hub: its sides are alternate hill and valley folds. Then, we draw n major fold lines, each forming an angle $\beta = \pi/n$ with a side of the polygon. Finally, we draw the n sets of equally spaced, parallel folds b and d, orthogonal to the sides of the hub.

2.1 Irregular hubs

There is no need for the hub to be a regular polygon, the only requirement —as explained earlier— is that n should be even. Obviously, equation (1) does not hold for irregular polygons, but equations (2 - 4) still hold. Note that the reasoning for equation (4) has to be modified slightly, but the final result does not change. Thus, given the inner hub angle α_i at vertex i, where i = 1, ... n, we have

$$\beta_i = \frac{\pi - \alpha_i}{2}, \text{ and } \delta_i = \frac{\pi}{2}, \tag{6}$$

from which the folds next to the hub can be drawn, as before. After that, since the n major folds are still straight, there is no difficulty to complete the fold pattern.

It is interesting to consider the case of an irregular polygon with equal sides but angles equal only at alternate vertices, e.g. for n = 4 a rhomboidal, instead of a square hub. In this case, any configuration has n/2-fold rotational symmetry. In the wrapped membrane the hub vertices are no longer at the centre, as in Fig. 1.

2.2 A regular, degenerate case

The idea of having alternate angles of different size can be taken to the limit. For example, let us start from the pattern of Fig. 5(a) and, while keeping the side lengths equal, let us change the inner

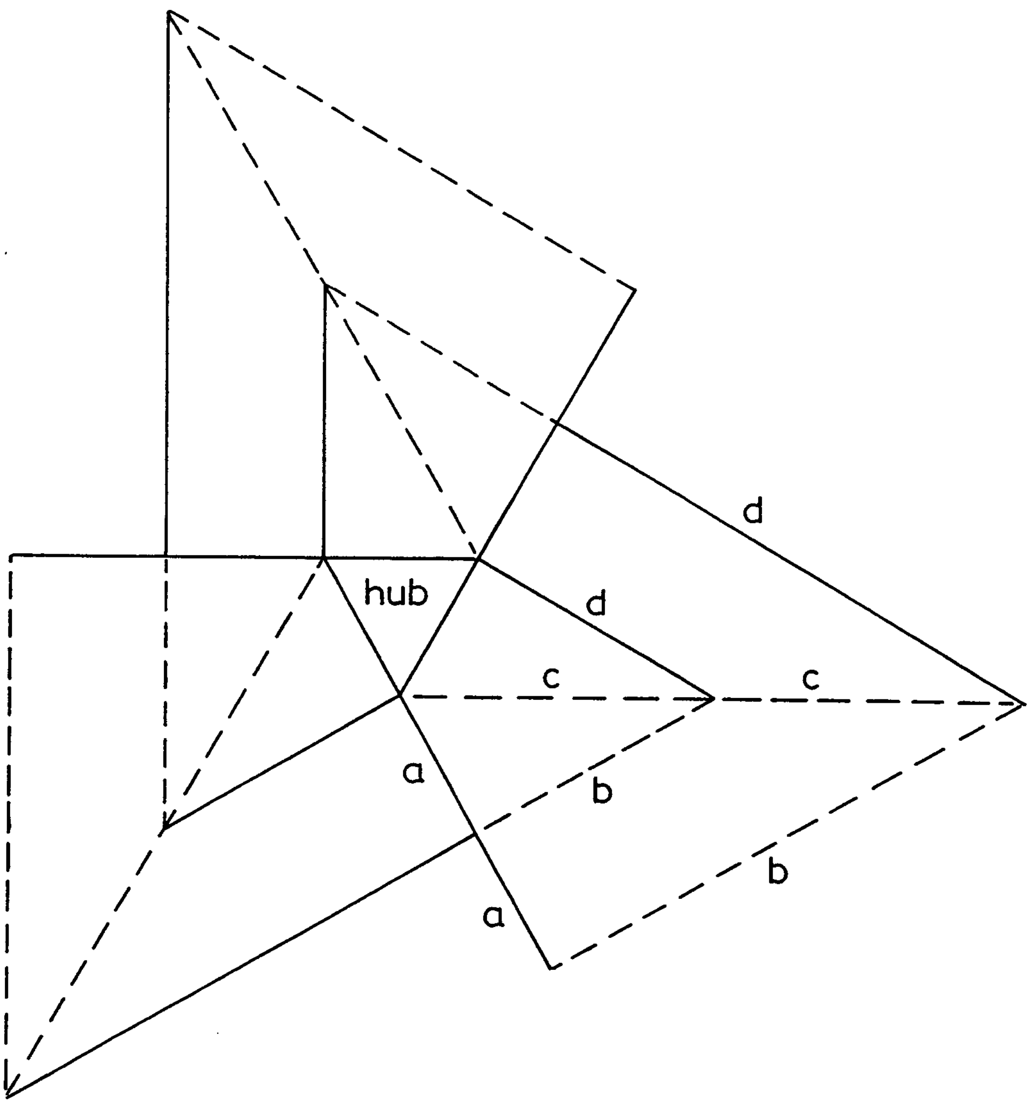


Fig. 6. Degenerate version of fold pattern in Fig. 5(a). Valley folds shown by dashed lines.

angles: $\alpha_1 = \alpha_3 = \alpha_5 \rightarrow \pi/3$ while $\alpha_2 = \alpha_4 = \alpha_6 \rightarrow \pi$. From equation (6) $\beta_1 = \beta_3 = \beta_5 \rightarrow \pi/3$ and $\beta_2 = \beta_4 = \beta_6 \rightarrow 0$. Thus, in the limit, we obtain the fold pattern shown in Fig. 6 with three major hill folds and three major valley folds, as the pattern from which it originated.

The fold pattern of Fig. 6 has three interesting features. First, in the wrapped configuration the hub is at the bottom of the membrane. Second, the vertices on the triangular hub are points of intersection of five folds, not just four as in the standard case. It might be expected that such degenerate arrangements would have more inextensional mechanisms than the standard fold pattern. However, it can be verified by means of the extended Maxwell's rule (Calladine, 1978) applied to a triangulated bar-and-joint model of the membrane that, actually, there are still only three independent mechanisms, one of which is activated for folding. Third, the central part of the folding pattern in Scheel (1974) is essentially identical to this one, although Fig. 3 has six hub vertices while Fig. 6 has only three and folds b and d are not shown in Fig. 3. We have just shown that these fold patterns require a polygonal hub, hence obviously a membrane folded according to Scheel's pattern will wrinkle in the region near the cylindrical hub. Furthermore, the major folds are straight, as in Figs 3 and 6, only for membranes of zero thickness, hence Scheel's pattern would not be correct in practice.

Although obtained as a special, degenerate case of the fold pattern of Fig. 5(a), the pattern of Fig. 6 —and many others with the same key feature of having five folds meeting at each hub vertex— could be obtained directly from an approach similar to that in Section 2. There is only one key difference: because at each vertex there are five angles but only four geometrical conditions on them, now there is some freedom and hence several different wrapped configurations can be obtained. The pattern of Fig. 6 satisfies the additional condition that two angles external to the hub are equal to $\pi/2$, hence the wrapped membrane has a flat lower edge, level with the hub. Many other choices are possible; for example, the lower edge of the membrane could be made to lie on a conical surface, entirely above the hub.

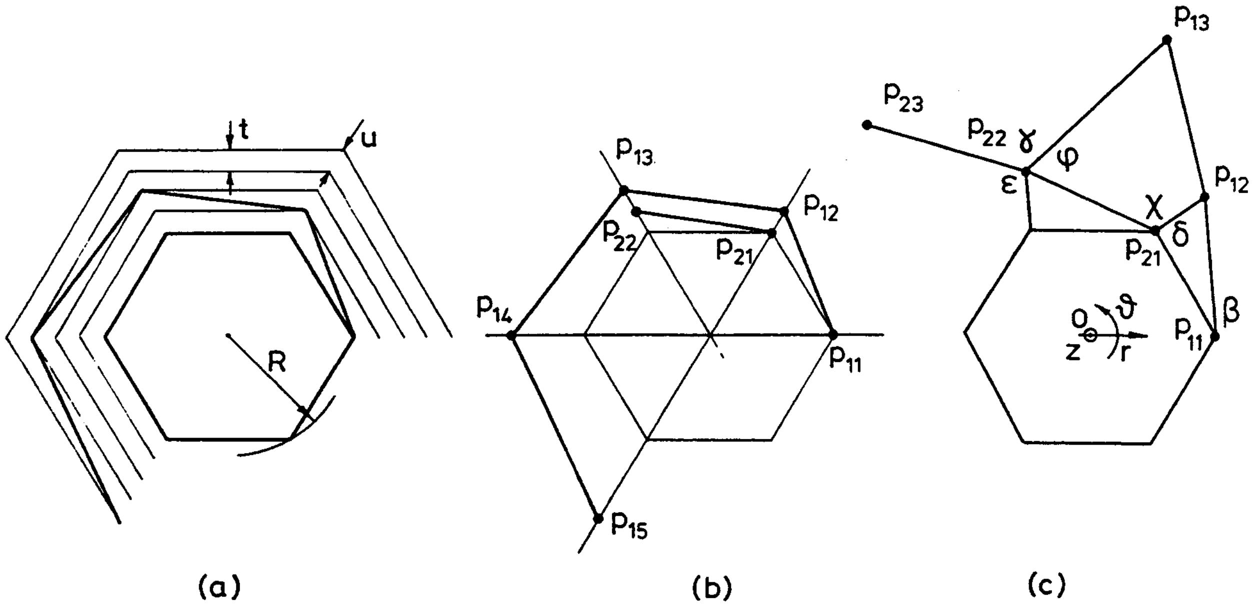


Fig. 7. (a) Plan view of a major fold line in the wrapped configuration, for t > 0. Vertex points P_{ij} on major folds in (b) wrapped and (c) flat configurations. Note that in (c) P_{12} is no longer on a radial plane through P_{12}, etc.

3. THIN MEMBRANES

The fold patterns derived in the previous section can be modified to account for a small membrane thickness $t > 0$. For the sake of simplicity, we shall consider only the basic fold pattern, based on regular polygonal hubs, as in Fig. 5(a). Thus all of the symmetry considerations in Section 2 are still valid. Our approach could be readily extended to irregular hubs and also to degenerate cases.

Obviously, the wrapped membrane will no longer coincide with the edge of the hub, a consideration which greatly simplified our analysis in Section 2. To make progress it is crucial that, having removed our earlier, unrealistic assumption that $t = 0$, we identify some key property that avoids unnecessary complications in the analysis. A rather obvious line of attack would be to think of the membrane as a series of thick, rigid panels hinged at the edges but this leads to fairly intractable singularities at the fold lines.

A more productive approach is to assume that the wrapping of a membrane with $t > 0$ around an n-sided polygon is essentially equivalent to the wrapping of a membrane with $t = 0$ such that, after folding, its vertices lie on helical curves whose radius increases at a constant rate, based on t. With this approach we neglect the (small) out-of plane bending of each membrane panel and also the localised bending near each vertex. Although of negligible importance in an overall sense, the deformation and stress state in the membrane near a vertex may need to be analysed separately.

Figure 7(a) shows the first major fold after wrapping, in plan view; if $t = 0$ it would coincide, of course, with the edge of the hub. Because the profiles of the other five major fold lines have identical shape, in plan view, they are obtained simply by rotation of the first fold about the centre of the hub. Therefore, membrane panels such as BCED in Fig. 5(c) are no longer plane, after wrapping, because BC and DE are not vertical, although they still lie in vertical, radial planes.

We shall now describe the process by which, starting from the hub, we can position the fold lines next to it and hence the first set of n vertices away from the hub. The next set of fold lines and vertices are positioned by the same process, and so on. Let us denote by P_{ij} the j-th vertex on the i-th major fold. For convenience, we shall work in the cylindrical coordinate system O, r, θ, z defined in Fig. 7(c). In the wrapped configuration, the vertices on the first major fold, assumed to be a hill fold, are

$$P_{11} = \begin{pmatrix} R \\ 0 \\ 0 \end{pmatrix}, \quad P_{12} = \begin{pmatrix} R + u \\ 2\pi / n \\ z_2 \end{pmatrix}, \quad P_{13} = \begin{pmatrix} R + 2u \\ 4\pi / n \\ z_3 \end{pmatrix}, \quad \text{etc.} \tag{7}$$

where z_2, z_3, etc. are positive quantities, to be determined, and

$$u = \frac{t}{\cos(\pi / n)} \quad . \tag{8}$$

For the second major fold, which is a valley fold:

$$P_{21} = \begin{pmatrix} R \\ 2\pi / n \\ 0 \end{pmatrix}, \; P_{22} = \begin{pmatrix} R + u \\ 4\pi / n \\ -z_2 \end{pmatrix}, \; P_{23} = \begin{pmatrix} R + 2u \\ 6\pi / n \\ -z_3 \end{pmatrix}, \text{ etc.} \tag{9}$$

Note that, for simplicity, the thickness t of the hub has been neglected and hence the z-coordinates of all vertices P_{i1}'s vanish. Also, by symmetry, the z-coordinates of equal numbered vertices on different folds are equal in absolute value; of course, this would not be true for irregular hubs.

Thus, the r, θ coordinates of all vertices as well as the z-coordinates of hub vertices are known in the wrapped configuration, but the z-coordinates of non-hub vertices are unknown. However, in Section 2 we had a condition on the angular defect at each point of the membrane by which these unknown coordinates can be found. The condition involves the four angles at any vertex. We can calculate the direction cosines of these angles by taking the dot-products of unit vectors between suitably chosen vertices, in the wrapped configuration. For example, Fig. 7(c), the angles at a hub vertex are

$$\beta = \arccos\left(\frac{(P_{12} - P_{11})}{\|P_{12} - P_{11}\|} \cdot \frac{(P_{21} - P_{11})}{\|P_{21} - P_{11}\|} \right) \tag{10}$$

$$\chi = \arccos\left(\frac{(P_{22} - P_{21})}{\|P_{22} - P_{21}\|} \cdot \frac{(P_{12} - P_{21})}{\|P_{12} - P_{21}\|} \right) \tag{11}$$

$$\delta = \arccos\left(\frac{(P_{21} - P_{11})}{\|P_{21} - P_{11}\|} \cdot \frac{(P_{12} - P_{21})}{\|P_{12} - P_{21}\|} \right) \tag{12}$$

while the angle α is still given by equation (1). The dot-products can easily be taken after converting the vertex cylindrical coordinates to a cartesian system with the same origin O. We can substitute these expressions into equation (2), then solve for the unknown z_2, and then calculate β, χ from equations (10, 11). Hence, we can draw the fold pattern as far as vertex P_{i2} for each major fold.

Next, to calculate z_3 and hence ε, ϕ, see Fig. 7(c), we express ε, ϕ, γ in terms of vertex coordinates by equations analogous to (10-12) and then substitute them into the angular defect condition for vertex P_{22}

$$\varepsilon + (\pi - \beta - \delta) + \phi + \gamma = 2\pi . \tag{13}$$

In the Appendix we give the fold patterns, computed as described above and then plotted by computer, for the following two cases. (i) A hexagonal hub and 2 mm thick membrane with five vertices on each major fold. This pattern is not meant for use with a 2 mm thick sheet (with R/t = 7.5 it would be far from thin) but with ordinary photocopying paper, to produce a fairly open wrapped configuration, suitable for simple demonstrations. (ii) A 12-sided hub and 0.2 mm thick membrane. If these patterns are magnified by, e.g., photocopying, note that the corresponding values of t are also magnified.

Folding patterns for irregular hubs can be computed by an approach similar to that described above but, because the z-coordinates of corresponding vertices are no longer equal in magnitude, a system of up to n equations in the z-coordinates of these vertices has to be set up and solved, instead of a single equation, as above.

4. DISCUSSION

A distinctive feature of the approach presented in this paper is that straight-sided polygonal hubs, not circular hubs, have been considered. Although the same assumption appears to have been made implicitly by all those who have worked on this problem before, because the fold patterns that we generate are basically in agreement with those drawn previously, we believe that this feature had not been fully exploited before. We think that continuously curved hubs are, almost certainly, ruled out. It might be argued that our approach could produce a circular hub by increasing the number of sides of a regular polygon, as $n \to \infty$. This would require infinitely many hill and valley folds, obviously impossible to arrange in practice. The wrapped membrane would have negligible height and very large thickness but, actually, the basic assumptions of the derivation, namely of no wrinkling and inextensional behaviour, would cease to be valid long before such large values of n are attained.

The main difficulty with curved folds can be explained as follows. A curved fold line in a flat membrane requires that the membrane be curved in opposite directions, since at any point of the curved fold two of the four folds required according to Miura (1989) are along the curved fold itself. These two folds have equal sign, hence two more folds of opposite signs are required. If the hub has to stay flat, as assumed above, then the outside region has to take a negative gaussian curvature, which requires some extensional deformation. It would be interesting to explore further this situation by a numerical approach (Miller et al., 1985).

By starting from the simpler and yet somewhat impractical case of zero-thickness membranes, we have been able to identify some key features of the fold patterns. Thus, it has been shown in Section 2 that the basic pattern, with four folds meeting at each hub vertex, is fully determined by the shape of the hub. Additional freedom is available in patterns where five folds meet at each hub vertex. This is a topic for further investigation.

As a final remark, we should like to emphasise that all fold patterns described in this paper have been derived from consideration of the flat and the wrapped configurations only. Hence, within the approximations of our approach, everything will fit in these two configurations. On this basis alone, it would be by no means certain that a continuous, inextensional transition from one state to the other is possible. At this stage, the only evidence that this is indeed the case is that many models that we have made work well and are apparently undamaged after many months of use. This is not an entirely satisfactory statement, of course, and further analysis of the deployment process is planned.

ACKNOWLEDGEMENTS

We thank Steve Temple for helpful discussions and for giving us one of his models. This work was partially supported by B.Ae. (Space Systems) Ltd, Stevenage. S.P. gratefully acknowledges the award of an ESA fellowship.

REFERENCES

Calladine, C. R., 1978, "Buckminster Fuller's Tensegrity structures and Clerk Maxwell's rules for the construction of stiff frames", International Journal Solids Structures, Vol. 14, pp 161-172.

Calladine, C.R., 1983, "Theory of shell structures", Cambridge University Press.

Cambridge Consultants, 1989, "Design study for a Mars spacecraft", Technical Report.

Huso, M. A., 1960, "Sheet reel", U.S. Patent no 2942794.

Johnson, W. and Yu, T. X., 1980, "The angle of fold and the plastic work done in the folding of developable flat sheets of metal", Journal of Mechanical Engineering Science, Vol. 22, pp 233-241.

Lanford, W. E., 1961, "Folding apparatus", U.S. Patent no 3010372.

Miller, R. K., Hedgepeth, J. M., Weingarten, V. I., Das, P. and Kahyai, S., 1985, "Finite element analysis of partly wrinkled membranes", Computers & Structures, Vol. 20, pp 631-639.

Miura, K., 1980, "Method of packaging and deployment of large membranes in space", Paper no IAF-80-A31, Proc. 31st IAF Congress, Tokyo.

Miura, K., Sakamaki, M., Suzuki, K., 1980, "A novel design of folded map", Congress of the International. Cartographical Association, Tokyo.

Miura, K., 1989, "A note on intrinsic geometry of Origami", Proc. International Meeting on Origami Science and Technology, Ferrara, Italy.

Scheel, H. W., 1974, " Space-saving storage of flexible sheets", U.S. Patent no 3848821.

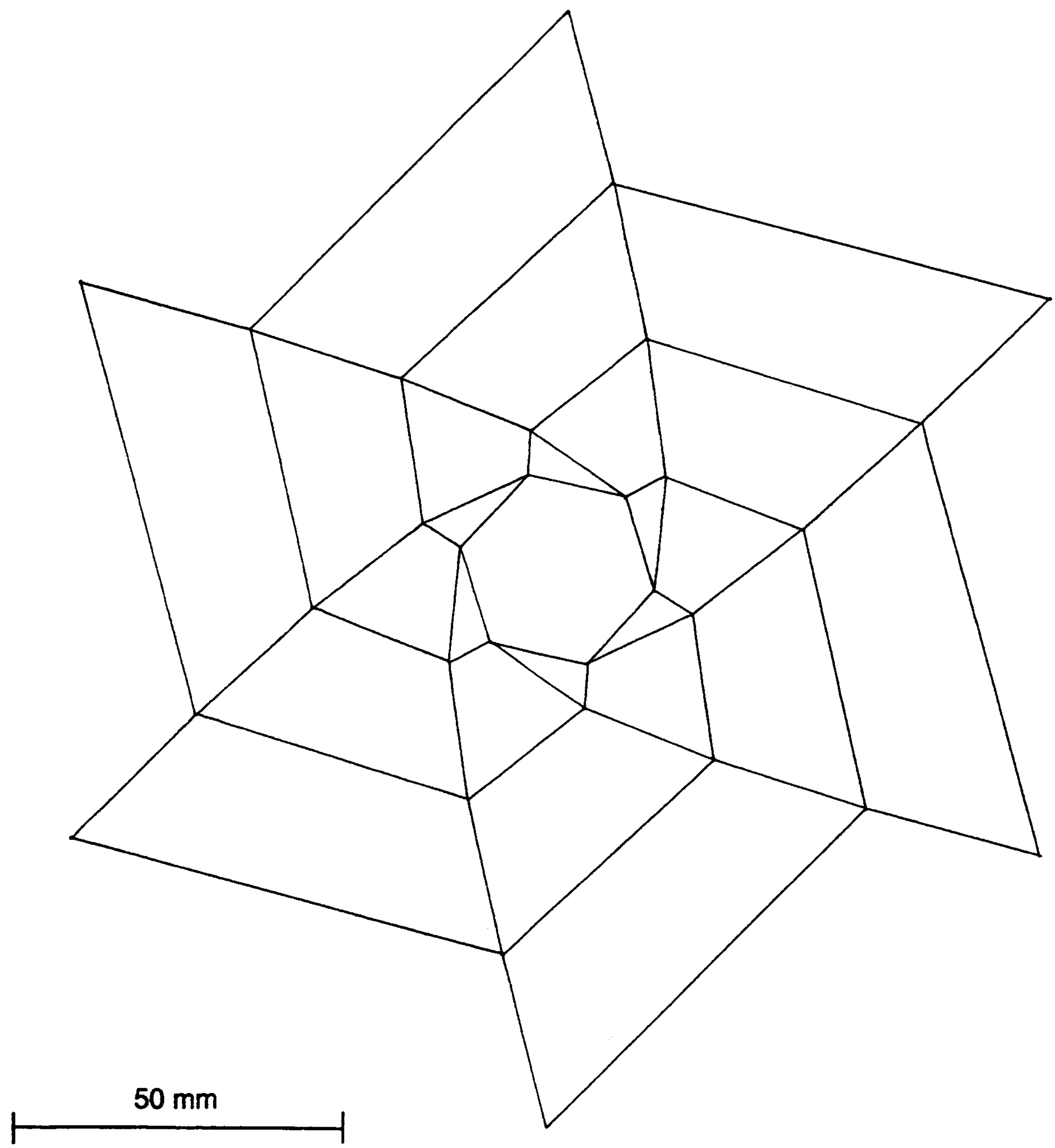
Wade, W. D. and McKean, V. C., 1981, "The technology development methodology for a class of large diameter space borne deployable antennas", Proc. 15th Aerospace Mechanisms Symposium, Marshall Space Flight Center, Alabama, pp 159-172.

Wright, J.L., 1991, "Space Sailing", Gordon and Breach Science Publishers.

APPENDIX

These patterns are suitable for photocopying (but watch out for distortions introduced by the copying machine) and producing simple paper models. It is suggested that fold lines be lightly scored with a blade on the outside of each fold, prior to folding. Extra care is required in the folding of the n = 12 model, hence it is better to begin with the first model to gain some practice.

n = 6, t = 2 mm



n = 12, t = 0.2 mm