In: New Approaches to Structural Mechanics, Shells and Biological Structures, eds. Drew, H.R., Pellegrino, S., Kluwer, 2002

SYMMETRY ANALYSIS OF THE DOUBLE BANANA AND RELATED INDETERMINATE STRUCTURES

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Abstract Symmetry arguments characterise the mechanisms and states of self-stress of the 'double banana' and its n-fold generalisations (bar-and-joint assemblies of n distorted bipyramids linked at shared apical vertices). The scalar form of Maxwell's rule shows only that any states of self-stress and mechanisms of these systems are equal in number, whereas the symmetry extension of the rule shows further that they are either equisymmetric (for odd n) or differ in two specific one-dimensional representations (for even n). General symmetry criteria are stated for the indeterminacy produced by similar condensations of rigid frameworks with suppression of two joints.

Keywords: Symmetry, rigidity, mechanism, self-stress

1. INTRODUCTION

The 'double banana' (Figure 1) is a classic example of a structure that satisfies Maxwell's rule for the rigidity of frames and yet is clearly indeterminate, since the two banana units are free to rotate about the line joining their common points (the so-called 'implied edge' (Graver, Servatius and Servatius, 1993)). It has been shown (Fowler and Guest, 2000) that a symmetry extension of Maxwell's rule can provide extra information that detects and characterises such indeterminancies in favourable

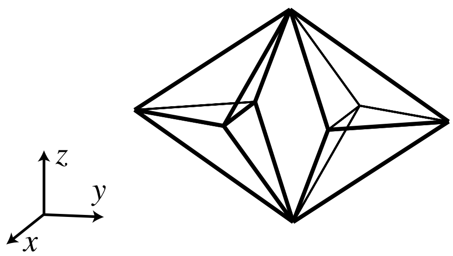

Figure 1. The 'double banana'.

cases. Here we show that the application of symmetry analysis of this type detects the mechanism and its associated state of self-stress in the double banana. A full symmetry description of mechanisms and states of self stress is also presented for the generalised n-fold 'multiple banana', and for some related frameworks produced by augmentation of polyhedra with banana subunits on all edges.

2. THE DOUBLE BANANA

Maxwell's rule (Maxwell, 1864), that a statically and kinematically determinate framework with b bars and j joints obeys

$$b = 3j - 6, \tag{1}$$

is clearly satisfied by the single 'banana' formed by face-fusion of two tetrahedra, which has $b = 9$, $j = 5$. When the double banana is created by joining two such units, with net elimination of two vertices, Figure 1, the Maxwell rule is still satisfied ($b = 18$, $j = 8$) but the composite structure now has an obvious mechanism. The two bananas can rotate freely about the line joining their common vertices. Calladine's statement of an extended Maxwell rule (Calladine, 1978),

$$s - m = b - 3j + 6, \tag{2}$$

where s and m count states of self-stress and mechanisms, respectively, implies the existence of a corresponding state of self-stress. This is easily seen to be the state in which one banana is in tension and the other is in compression. Counting structural components alone cannot deduce $s = m = 1$ from $s - m = 0$, but in this case, consideration of component symmetries can.

The symmetry extension of Maxwell's rule (Fowler and Guest, 2000) is

$$\Gamma(s) - \Gamma(m) = \Gamma(b) - [\Gamma(j) \times \Gamma_T - \Gamma_T - \Gamma_R], \tag{3}$$

where $\Gamma(s)$, $\Gamma(m)$ etc. are the representations of s states of self-stress, m mechanisms, b bars and j joints, and Γ_T, Γ_R are the translational and rotational representations, all within the point group of the undistorted frame. Several examples of the use of this formula are given in the original reference.

In the present case, the point group of the double banana is D_{2h} and the evaluation of the term in square brackets on the RHS of (3) proceeds as

D_{2h}	E	$C_2(z)$	$C_2(y)$	$C_2(x)$	i	$\sigma(xy)$	$\sigma(xz)$	$\sigma(yz)$
$\Gamma(j)$	8	2	2	0	0	6	2	4
$\times\Gamma_T$	3	−1	−1	−1	−3	1	1	1
=	24	−2	−2	0	0	6	2	4
$-\Gamma_T - \Gamma_R$	−6	2	2	2	0	0	0	0
=	18	0	0	2	0	6	2	4

which reduces to

$$4A_g + 2B_{1g} + B_{2g} + 2B_{3g} + A_u + 2B_{1u} + 3B_{2u} + 3B_{3u}.$$

Similarly, the first term on the RHS of (3) gives

D_{2h}	E	$C_2(z)$	$C_2(y)$	$C_2(x)$	i	$\sigma(xy)$	$\sigma(xz)$	$\sigma(yz)$
$\Gamma(b)$	18	0	2	0	0	6	0	6

which reduces to

$$4A_g + 2B_{1g} + B_{2g} + 2B_{3g} + A_u + 2B_{1u} + 4B_{2u} + 2B_{3u}.$$

Hence, by subtraction, $\Gamma(s) - \Gamma(m) = B_{2u} - B_{3u}$, and the separate symmetries of the state of self-stress and the mechanism are $\Gamma(s) = B_{2u}$ and $\Gamma(m) = B_{3u}$, respectively.

B_{2u} is the symmetry of a vector pointing along the y axis (Atkins, Child and Phillips, 1970), and matches the dipolar state of self-stress in which the two bananas have equal and opposite bar forces. Likewise, B_{3u} is the symmetry of a vector pointing along the x axis, and matches the mechanism, in which the two bananas have equal and opposite rotational displacements about the common axis. Figure 2 represents these symmetries in a symbolic notation which will be useful for larger systems.

This linearised analysis shows the existence of a local, possibly infinitesimal, possibly finite, mechanism. In the special case of the double banana it is possible to deduce that the mechanism is in fact finite. This

(a) (b)

Figure 2. Representation of (a) the single state of self-stress, and (b) the single mechanism, of the double banana. Each vertex represents a banana: in (a), the + or − represents an overall state of tension, or compression; in (b), the arrow represents rotation of the banana around the shared implied edge.

follows from the fact that the symmetries of the mechanism and the state of self-stress remain distinct along the path followed by the mechanism (Guest and Kangwai, 1999), which preserves C_{2v} symmetry. In this lower group, the mechanism is totally symmetric, whereas the state of self-stress is antisymmetric under the C_2 rotation and one reflection operation.

Although the banana unit considered here has a triangular cross-section, the analysis is easily seen to generalise to bipyramidal units based on any cross-section, including the pentagon of the biological banana, as long as the two units remain symmetry-equivalent under a reflection or C_2 rotation. The following section shows how far these considerations may be extended for systems with more than two banana subunits.

3. THE MULTIPLE BANANA

An obvious generalisation of the double banana is the *multiple banana*: a set of n bananas in which exactly two joints are common to all n subunits (Figure 3). The multiple banana satisfies Maxwell's rule (with $b = 9n$, $j = 3n + 2$) and obeys Calladine's extended version with $s = m$, but it is also intuitively clear that $m = n - 1$ (each banana is free to rotate about the axis defined by the two common joints, but concerted rotation of all n constitutes a rigid-body motion rather than a mechanism), and hence $s = n - 1$ (tensions can be freely chosen for $n - 1$ subunits, that in the final subunit being constrained by the requirement of preserving equilibrium).

An analysis similar to that in the previous section can be performed for any particular value of n, but a more general formulation is also straightforwardly constructed. For this purpose, the full three-dimensional n-fold multiple banana framework can be replaced by an n-vertex polygon, where each vertex stands for a banana subunit. A local tension in a subunit is then shown as a scalar at the relevant vertex, and the rotational

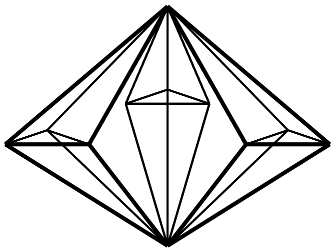

Figure 3. An example of the multiple banana, here a triple banana with three sub-units.

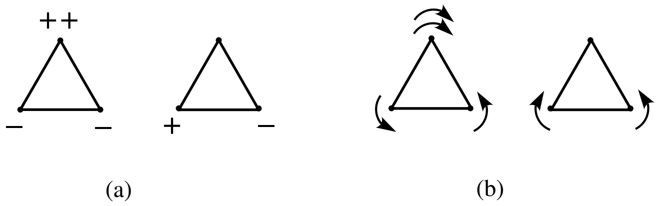

Figure 4. (a) The two states of self-stress, and (b) the two mechanisms, of the 3-banana, described using the notation introduced in Figure 2. A double + or arrow represents a quantity with double magnitude.

motion of a single subunit as a tangential vector at that vertex, Figure 4.

The representation of the $n - 1$ states of self-stress in the symmetry group that describes the undistorted structure (in this case D_{nh} for an n-fold banana) is then seen to be

$$\Gamma(s) = \Gamma_\sigma(v) - \Gamma_0 \tag{4}$$

where $\Gamma_\sigma(v)$ is the permutation representation of the vertices of the polygon, and Γ_0 is the totally symmetric representation, which has character $+1$ under every operation of the symmetry group. The interpretation of this group-theoretical formula is that the states of self-stress comprise all the independent combinations of local tensions (vertex scalars) that sum overall to zero, or in other words all those that are orthogonal to the totally symmetric $(1, 1, 1, \ldots)$ combination.

Likewise, the representation of the mechanisms of the n-fold multiple banana is

$$\Gamma(m) = \Gamma_\rightarrow(v) - \Gamma_{R_z} \tag{5}$$

where $\Gamma_\rightarrow(v)$ is the representation of a set of in-plane tangential vectors, one for each vertex, and Γ_{R_z} is the representation of a rotation about the axis of the structure. The relation (5) expresses the fact that any combination of such vectors, with the exception of the concerted rigid-rotation, stands for an internal mechanism.

$\Gamma_\sigma(v)$ is a familar object, tabulated in the chemical literature, but $\Gamma_\rightarrow(v)$ is more unusual. However, its evaluation can be side-stepped by use of the identity (Fowler and Quinn, 1986)

$$\Gamma_\sigma(v) \times \Gamma_T = \Gamma_\sigma(v) + \Gamma_\pi(v) \tag{6}$$

where $\Gamma_\pi(v)$ is the representation of the set of pairs of orthogonal tangential vectors at vertex sites. In systems with a horizontal mirror plane, the π space splits into disjoint representations of the in-plane and out-of-plane tangential vectors:

$$\Gamma_\pi(v) = \Gamma_\uparrow(v) + \Gamma_\rightarrow(v). \tag{7}$$

The part of $\Gamma_\pi(v)$ that is antisymmetric with respect to the horizontal plane is

$$\Gamma_\uparrow(v) = \Gamma_\sigma(v) \times \Gamma_{T_z} \tag{8}$$

where T_z is a translation along the principal axis, so that the desired symmetry $\Gamma_\rightarrow(v)$ can be found by subtraction as

$$\Gamma_\rightarrow(v) = \Gamma_\pi - \Gamma_\sigma(v) \times \Gamma_{T_z} = \Gamma_\sigma(v) \times \Gamma_{T_x,T_y} - \Gamma_\sigma(v), \tag{9}$$

where Γ_{T_x,T_y} is the representation of the in-plane translations T_x and T_y. It is therefore possible to express the symmetries of the mechanisms in terms of $\Gamma_\sigma(v)$ only:

$$\Gamma(m) = \Gamma_\sigma(v) \times \{\Gamma_{T_x,T_y} - \Gamma_0\} - \Gamma_{R_z}. \tag{10}$$

Equation (10) is completely general for all n, and, taken with (4), satisfies the target of giving an explicit expression for both $\Gamma(s)$ and $\Gamma(m)$ in terms of a property of the polygon (namely $\Gamma_\sigma(v)$), but it can be given a more transparent form if the two cases of odd and even n are treated separately. It turns out then that

$$(n \text{ odd}) \qquad \Gamma_\rightarrow(v) = \Gamma_\sigma(v) - \Gamma_0 + \Gamma_{R_z} \tag{11a}$$

and

$$(n \text{ even}) \qquad \Gamma_\rightarrow(v) = \Gamma_\sigma(v) \times \Gamma_{R_z}. \tag{11b}$$

Both statements are easily proved by considering characters under the possible symmetry operations of D_{nh} for odd and even n. It follows that

$$(n \text{ odd}) \qquad \Gamma(m) = \Gamma(s) \tag{12a}$$

and

$$(n \text{ even}) \qquad \Gamma(m) = \Gamma(s) \times \Gamma_{R_z}. \tag{12b}$$

Equations (4) and (12) give a complete symmetry specification of the indeterminacy of the multiple banana. From (12), the representation $\Gamma(s) - \Gamma(m)$ vanishes identically for odd n and for even n is given by

$$(n \text{ even}) \qquad \Gamma(s) - \Gamma(m) = \{\Gamma_\sigma(v) - \Gamma_0\} \times \{\Gamma_0 - \Gamma_{R_z}\}. \tag{13}$$

A further simplication is possible. Equation (13) apparently has a linear dependence on n, through $\Gamma_\sigma(v)$. In fact, $\Gamma_\sigma(v)$ can be eliminated altogether if we define a representation $\Gamma_\star$, which is the symmetry of a decoration of the vertices of the polygon with consistently alternating + and − signs all the way around the ring. In a chemical context $\Gamma_\star$ is a powerful tool in the study of unsaturated systems; its relevance follows from the bi-partite nature of the n-polygon, as the sign decoration can be achieved only for even values of n. In terms of the $\Gamma_\star$ representation,

$$(n \text{ even}) \qquad \Gamma(s) - \Gamma(m) = \Gamma_\star \times \{\Gamma_0 - \Gamma_{R_z}\}. \tag{14}$$

Equivalence of (13) and (14) follows from the angular momentum properties of $\Gamma_\sigma(v)$, which consists, apart from Γ_0 and $\Gamma_\star$ of pairs $\Gamma_{\pm\Lambda}$ that are self-conjugate under R_z and so make no net contribution to the RHS of (13). The equivalence can also be proved by direct comparison of characters.

In point groups C_{nv}, D_n, D_{nh} and D_{nd}, Γ_{R_z} is non-degenerate and distinct from Γ_0, and the direct inference from the negative sign on the RHS of (14) is that, for even n, $\Gamma(m)$ always contains the one-dimensional representation $\Gamma_\star \times \Gamma_{R_z}$, whereas $\Gamma(s)$ contains $\Gamma_\star$ itself.

The implications for the application of the Maxwell rule and its various extended forms to these systems can be summarised as four subcases.

$n = 1$. Both counting and symmetry versions of the rule agree in correctly predicting that $s = m = 0$.

$n = 2$. Simple counting fails, but the symmetry version correctly predicts $s = m = 1$ with $\Gamma(s) = B_{2u}$ and $\Gamma(m) = B_{3u}$.

odd $n > 1$. Counting correctly predicts $s = m$, and the symmetry version extends this to the equisymmetry $\Gamma(s) = \Gamma(m)$. Further considerations give $s = m = n - 1$ and $\Gamma(s) = \Gamma(m) = \Gamma_\sigma(v) - \Gamma_0$.

even $n > 2$. Counting correctly predicts $s = m$, and the symmetry version extends this to $\Gamma(s) - \Gamma(m) = \Gamma_\star - \Gamma_\star \times \Gamma_{R_z}$, with $\Gamma(s)$ containing $\Gamma_\star$ and $\Gamma(m)$ containing $\Gamma_\star \times \Gamma_{R_z}$. Further considerations fix the full sets of symmetries as $\Gamma(s) = \Gamma_\sigma(v) - \Gamma_0 = \Gamma(m) \times \Gamma_{R_z}$.

This completes the characterisation of the indeterminacy of the multiple banana. Similar reasoning can be extended to frameworks based on attachment of banana units to polyhedral frameworks, as will be shown in the next section.

4. OTHER AUGMENTED FRAMEWORKS

The reason for the celebrity in rigidity theory of the double banana is that condensation of two rigid frameworks introduces freedom in this simple system, without disturbing the Maxwell bar count. Any two determinate frameworks can be condensed in this way, across any pair of joints, whether linked by a bar or not, with consequent introduction of a degree of freedom. Condensation across pairs of joints directly linked by a bar can also be used to introduce additional freedoms in cases where the original frameworks were not themselves rigid. Multiple bananas are the result of repeated formal condensation of this last type, all based on the single implied edge of the double banana.

Another family of statically indeterminate bar-and-joint assemblies can be constructed when augmentation with banana subunits is carried out globally, once per bar, on a framework. Symmetry arguments can again give a helpful improvement over the simple Maxwell counting procedure for these augmented systems. Specifically, let the initial framework be a polyhedron P. Each additional subunit spanning an edge of P contributes an extra state of self-stress associated with a compression or tension that can be denoted by a scalar associated with that edge, and the total set of such states therefore has symmetry

$$\Gamma(s_{\text{extra}}) = \Gamma_\sigma(e), \tag{15}$$

i.e. the permutation representation of the edges (bars) of P. Each subunit also introduces a rotational freedom about the augmented edge, and the total set of such mechanisms spans the symmetry

$$\Gamma(m_{\text{extra}}) = \Gamma_\perp(e), \tag{16}$$

the representation of a set of tangential vectors across the edges of P. Thus the Maxwell indeterminacy symmetry for all such systems is the difference of a scalar and a vector representation defined on the polyhedron edges:

$$\Gamma(s_{\text{extra}}) - \Gamma(m_{\text{extra}}) = \Gamma_\sigma(e) - \Gamma_\perp(e). \tag{17}$$

Both $\Gamma_\sigma(e)$ and $\Gamma_\perp(e)$ can be expressed in terms of other structural representations (Ceulemans and Fowler, 1991), and in specialised forms when P belongs to the class of deltahedra, or alternatively of their duals, the trivalent polyhedra.

Three examples of the augmentation of initially determinate structures (so that $\Gamma(s) = \Gamma(s_{\text{extra}})$, $\Gamma(m) = \Gamma(m_{\text{extra}})$) are presented. Augmentation of a simple triangle leads to a 30-bar, 12-joint system that is completely specified by the symmetry-extended Maxwell rule. In the D_{3h} group, with z taken along the three-fold axis, evaluation of (17) gives

$$\Gamma(s) - \Gamma(m) = A_1' + E' - A_1'' - E'', \tag{18}$$

and hence

$$\begin{aligned} \Gamma(s) &= A_1' + E', \\ \Gamma(m) &= A_1'' + E'', \end{aligned} \tag{19}$$

as $\Gamma_\perp(e) = \Gamma_\sigma(e) \times \Gamma_{T_z} = \Gamma_\sigma(e) \times A_1''$ for the underlying triangle.

Augmentation of a tetrahedron leads to

$$\begin{aligned} \Gamma(s) &= A_1 + E + T_2, \\ \Gamma(m) &= T_1 + T_2, \end{aligned} \tag{20}$$

and as the s and m have a representation in common, their difference alone, in the symmetry-extended Maxwell rule, would give only partial information on each.

Similarly, augmentation of an octahedron yields a framework with

$$\begin{aligned} \Gamma(s) &= A_{1g} + E_g + T_{2g} + T_{1u} + T_{2u}, \\ \Gamma(m) &= A_{2u} + E_u + T_{1g} + T_{2g} + T_{1u}, \end{aligned} \tag{21}$$

and again the pair of representations carry more information than the Maxwell difference alone.

5. CONCLUSION

The aim of the present paper has been to confirm the utility of symmetry arguments as an adjunct to static and kinetic analysis of frameworks. Symmetry extension of the Maxwell rule is sufficient to provide a full description of the double banana. The outcome of the detailed analysis in Section 3 is an essential symmetry distinction between odd-n and even-n multiple-banana systems. When n is odd, the mechanisms and states of self-stress are equisymmetric; when n is even, their symmetries differ only by two (one-dimensional) representations. This distinction further defines the amount of information that can be gained from the symmetry-extended Maxwell equation in such cases.

Finally, it was noted that the multiple banana is only one, highly symmetric example of a more general class of problem. Any case where linkage of two fully determinate frameworks is carried out by identifying a pair of vertices introduces equal numbers of states of self-stress and balancing mechanisms, as does each repetition of the operation.

Symmetry considerations of the structural representations $\Gamma_\sigma(v)$, $\Gamma_\sigma(e)$, and their vector derivatives will identify completely $\Gamma(s)$ and $\Gamma(m)$ for augmented frameworks, and the symmetry-extended Maxwell rule will improve on the counting version in many cases. The degree of advantage will of course depend on the amount of symmetry in the structure, but where such symmetries exist, their incorporation in the analysis will always be fruitful.

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