Supplementary Material of the manuscript
Prestrain-induced bistability in the design of tensegrity units for mechanical metamaterials

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I. SIX-NODE UNIT CALCULATIONS

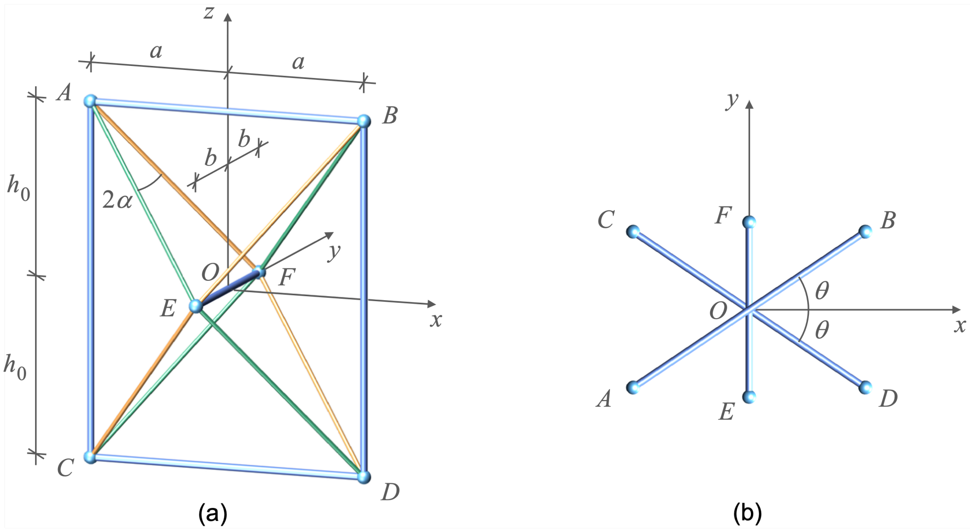

FIG. S1. The six-node tensegrity unit: (a) at the configuration with D_{2h} symmetry, axonometric view; (b) at a configuration with D_2 symmetry, projection onto the x-y plane with only bars AB, CD, and EF shown. The parameter θ defines the configuration in the single-DOF model.

We make use of the formula

$$|PQ|^2 = r_P^2 + r_Q^2 - 2\,r_P r_Q \cos(\varphi_P - \varphi_Q) + (z_P - z_Q)^2$$

for computing the distance between two points in the cylindrical coordinate system, $\{r, \varphi, z\}$ centered on the vertical symmetry axis.

We define $2h(\theta) = z_A - z_C = z_B - z_D$, with $h(0) = c$. We have

$$|AD|^2 = (2c)^2 = (2h(\theta))^2 + 2a^2(1 - \cos(2\theta)), \tag{S1}$$

so that

$$h^2(\theta) = c^2 + \frac{a^2}{2}(\cos(2\theta) - 1). \tag{S2}$$

As to the lengths of the springs, we have

$$\lambda_1^2(\theta) = a^2 + b^2 - 2ab\cos\left(\frac{\pi}{2} + \theta\right) + h^2(\theta), \tag{S3}$$

$$\lambda_2^2(\theta) = a^2 + b^2 - 2ab\cos\left(\frac{\pi}{2} - \theta\right) + h^2(\theta), \tag{S4}$$

substituting (S2) we obtain

$$\lambda_1(\theta)^2 = c^2 + b^2 + \frac{a^2}{2}(1+\cos(2\theta)) - 2ab\cos\left(\frac{\pi}{2}+\theta\right); \tag{S5}$$

$$\lambda_2(\theta)^2 = c^2 + b^2 + \frac{a^2}{2}(1+\cos(2\theta)) - 2ab\cos\left(\frac{\pi}{2}-\theta\right). \tag{S6}$$

The first and second derivatives of these quantities are

$$2\lambda_1\lambda_1' = -a^2\sin(2\theta) + 2ab\cos\theta, \tag{S7}$$

$$2\lambda_2\lambda_2' = -a^2\sin(2\theta) - 2ab\cos\theta, \tag{S8}$$

$$2\lambda_1\lambda_1'' = -2a^2\cos(2\theta) - 2ab\sin\theta - 2(\lambda_1')^2, \tag{S9}$$

$$2\lambda_2\lambda_2'' = -2a^2\cos(2\theta) + 2ab\sin\theta - 2(\lambda_2')^2. \tag{S10}$$

The potential energy of the system is given by the elastic energy

$$U(\theta) = 2k\Big((\lambda_1 - \overline{\lambda})^2 + (\lambda_2 - \overline{\lambda})^2\Big), \tag{S11}$$

in which $\overline{\lambda}$ is the common rest-length of the springs. We compute the first derivative and set it equal to zero to find the stationary points; we have:

$$U'(\theta) = 4k\Big(\lambda_1'(\lambda_1 - \overline{\lambda}) + \lambda_2'(\lambda_2 - \overline{\lambda})\Big) = 0, \tag{S12}$$

obtaining the condition

$$\lambda_1'(\lambda_1 - \overline{\lambda}) = -\lambda_2'(\lambda_2 - \overline{\lambda}). \tag{S13}$$

We see that $\theta = 0$ is a stationary point for the energy, since

$$\lambda_1(0)^2 = \lambda_2(0)^2 = a^2 + b^2 + c^2 =: \lambda_0^2, \tag{S14}$$

and

$$\lambda_1'(0) = \frac{ab}{\lambda_0} = -\lambda_2'(0). \tag{S15}$$

The second derivative of the energy is given by

$$U''(\theta) = 4k\Big(\lambda_1''(\lambda_1 - \overline{\lambda}) + \lambda_2''(\lambda_2 - \overline{\lambda}) + (\lambda_1')^2 + (\lambda_2')^2\Big). \tag{S16}$$

Since

$$\lambda_1''(0) = \lambda_2''(0) = -\frac{a^2}{\lambda_0}\left(1 + \frac{b^2}{\lambda_0^2}\right), \tag{S17}$$

we find that

$$U''(0) = 8k\frac{a^2}{\lambda_0}\left(-\left(\lambda_0 + \frac{b^2}{\lambda_0}\right)\frac{(\lambda_0 - \overline{\lambda})}{\lambda_0} + \frac{b^2}{\lambda_0}\right). \tag{S18}$$

After introducing the initial strain (prestrain) ε_0,

$$\varepsilon_0 := \frac{\lambda_0 - \overline{\lambda}}{\lambda_0}, \tag{S19}$$

we require that

$$U''(0) > 0, \tag{S20}$$

obtaining

$$-\left(\lambda_0 + \frac{b^2}{\lambda_0}\right)\varepsilon_0 + \frac{b^2}{\lambda_0} > 0. \tag{S21}$$

By considering that $\frac{b^2}{\lambda_0^2} = \sin^2\alpha$, with $\alpha = \frac{1}{2}\widehat{EAF}$, this condition can be rewritten as

$$\varepsilon_0 < \frac{1}{1 + \dfrac{1}{\sin^2\alpha}} =: \varepsilon_{\text{crit}}, \tag{S22}$$

II. EIGHT-NODE UNIT CALCULATIONS

The angles θ_1 and θ_2 are the two Lagrangian parameters for the system. At any given configuration, we have

$$(2h_c(\theta_1))^2 = (2c)^2 - 2a^2(1 - \cos 2\theta_1); \tag{S23}$$

$$(2h_d(\theta_2))^2 = (2d)^2 - 2b^2(1 - \cos 2\theta_2), \tag{S24}$$

with

$$h_c(0) = c, \qquad h_d(0) = d. \tag{S25}$$

We compute the first and second derivatives of these quantities:

$$h_c h_c' = -\frac{a^2}{2}\sin 2\theta_1; \tag{S26}$$

$$h_d h_d' = -\frac{b^2}{2}\sin 2\theta_2; \tag{S27}$$

$$h_c h_c'' = -a^2\cos 2\theta_1 - h_c'^2; \tag{S28}$$

$$h_d h_d'' = -b^2\cos 2\theta_2 - h_d'^2; \tag{S29}$$

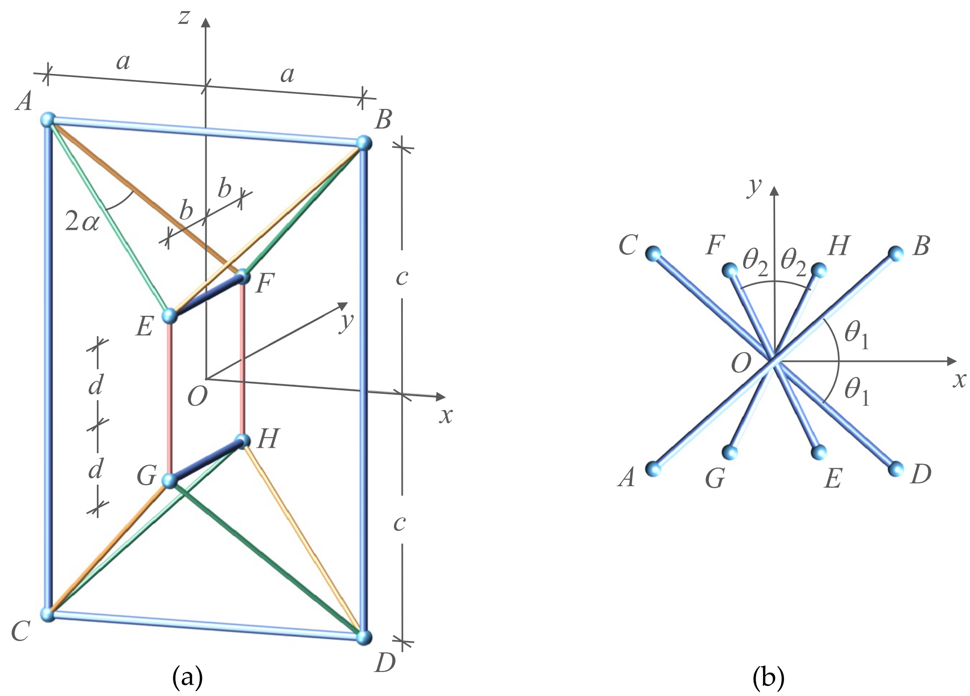

FIG. S2. The eight-node tensegrity unity: (a) at the configuration with D_{2h} symmetry, axonometric view; (b) at a configuration with D_2 symmetry, projection onto the x-y plane with only bars AB, CD, EF, and GH shown. The parameters θ_1 and θ_2 define the configuration in the two-DOF model.

In particular, for $(\theta_1, \theta_2) = (0,0)$ we have:

$$h_c'(0) = h_d'(0) = 0; \tag{S30}$$

$$h_c''(0) = -\frac{a^2}{c}; \tag{S31}$$

$$h_d''(0) = -\frac{b^2}{d}; \tag{S32}$$

As to the lengths of the springs, they are given by

$$\lambda_1^2(\theta_1, \theta_2) = a^2 + b^2 + (h_c(\theta_1) - h_d(\theta_2))^2 - 2ab\cos\left(\frac{\pi}{2} + \theta_2 - \theta_1\right); \tag{S33}$$

$$\lambda_2^2(\theta_1, \theta_2) = a^2 + b^2 + (h_c(\theta_1) - h_d(\theta_2))^2 - 2ab\cos\left(\frac{\pi}{2} - \theta_2 + \theta_1\right). \tag{S34}$$

Again, we compute the partial derivatives of these quatities:

$$2\lambda_1\lambda_{1,1} = 2(h_c - h_d)h_c' - 2ab\cos(\theta_2 - \theta_1);$$

$2\lambda_1\lambda_{1,11} = 2(h_c - h_d)h_c'' + 2h_c'^2 - 2ab\sin(\theta_2 - \theta_1) - 2(\lambda_{1,1})^2;$

$2\lambda_1\lambda_{1,12} = -2h_c'h_d' + 2ab\sin(\theta_2 - \theta_1) - 2\lambda_{1,2}\lambda_{1,1};$

$2\lambda_1\lambda_{1,2} = -2(h_c - h_d)h_d' + 2ab\cos(\theta_2 - \theta_1);$

$2\lambda_1\lambda_{1,22} = -2(h_c - h_d)h_d'' + 2h_d'^2 - 2ab\sin(\theta_2 - \theta_1) - 2(\lambda_{1,2})^2;$

$2\lambda_2\lambda_{2,1} = 2(h_c - h_d)h_c' + 2ab\cos(-\theta_2 + \theta_1);$

$2\lambda_2\lambda_{2,11} = 2(h_c - h_d)h_c'' + 2h_c'^2 - 2ab\sin(-\theta_2 + \theta_1) - 2(\lambda_{2,1})^2;$

$2\lambda_2\lambda_{2,12} = -2h_c'h_d' + 2ab\sin(-\theta_2 + \theta_1) - 2\lambda_{2,2}\lambda_{2,1};$

$2\lambda_2\lambda_{2,2} = -2(h_c - h_d)h_d' - 2ab\cos(-\theta_2 + \theta_1);$

$2\lambda_2\lambda_{2,22} = -2(h_c - h_d)h_d'' - 2h_d'^2 - 2ab\sin(-\theta_2 + \theta_1) - 2(\lambda_{2,2})^2.$

For $(\theta_1, \theta_2) = (0,0)$ we have:

$$\begin{aligned}
&\lambda_1^2(0,0) = \lambda_2^2(0,0) = \lambda_0^2 = a^2 + b^2 + (c-d)^2;\\
&\lambda_{1,1}(0,0) = \lambda_{2,2}(0,0) = -\frac{ab}{\lambda_0};\\
&\lambda_{1,2}(0,0) = \lambda_{2,1}(0,0) = \frac{ab}{\lambda_0};\\
&\lambda_{1,11}(0,0) = \lambda_{2,11}(0,0) = \frac{1}{\lambda_0}\Big(-\frac{a^2}{c}(c-d) - \frac{a^2b^2}{\lambda_0^2}\Big);\\
&\lambda_{1,22}(0,0) = \lambda_{2,22}(0,0) = \frac{1}{\lambda_0}\Big(\frac{b^2}{d}(c-d) - \frac{a^2b^2}{\lambda_0^2}\Big);\\
&\lambda_{1,12}(0,0) = \lambda_{2,12}(0,0) = \frac{a^2b^2}{\lambda_0^3}.
\end{aligned} \tag{S35}$$

On denoting by $\overline{\lambda}$ the common rest-length of the springs, the elastic energy is given by

$$U(\theta_1, \theta_2) = 2k\Big((\lambda_1 - \overline{\lambda})^2 + (\lambda_2 - \overline{\lambda})^2\Big). \tag{S36}$$

Equilibrium configurations can be obtained as stationary points of the energy, by setting its partial derivatives equal to zero. We have:

$$U_{,1} = 4k\Big(\lambda_{1,1}(\lambda_1 - \overline{\lambda}) + \lambda_{2,1}(\lambda_2 - \overline{\lambda})\Big) = 0, \tag{S37}$$

$$U_{,2} = 4k\Big(\lambda_{1,2}(\lambda_1 - \overline{\lambda}) + \lambda_{2,2}(\lambda_2 - \overline{\lambda})\Big) = 0, \tag{S38}$$

where $(\cdot)_{,i}$ denotes the partial derivative with respect to θ_i $(i = 1,2)$. It is easy to see that $(\theta_1, \theta_2) = (0,0)$ is an equilibrium configuration. The second partial derivatives of the energy are

$$U_{,11} = 4k\Big(\lambda_{1,11}(\lambda_1 - \overline{\lambda}) + \lambda_{2,11}(\lambda_2 - \overline{\lambda}) + (\lambda_{1,1})^2 + (\lambda_{2,1})^2\Big), \tag{S39}$$

$$U_{,22} = 4k\Big(\lambda_{1,22}(\lambda_1 - \overline{\lambda}) + \lambda_{2,22}(\lambda_2 - \overline{\lambda}) + (\lambda_{1,2})^2 + (\lambda_{2,2})^2\Big), \tag{S40}$$

$$U_{,12} = 4k\Big(\lambda_{1,12}(\lambda_1 - \overline{\lambda}) + \lambda_{2,12}(\lambda_2 - \overline{\lambda}) + \lambda_{1,1}\lambda_{1,2} + \lambda_{2,1}\lambda_{2,2}\Big). \tag{S41}$$

For $(\theta_1, \theta_2) = (0,0)$ we have:

$$U_{,11}(0,0) = 8k\left(\Big(-\frac{a^2}{c}(c-d) - \frac{a^2b^2}{\lambda_0^2}\Big)\frac{\lambda_0 - \overline{\lambda}}{\lambda_0} + \frac{a^2b^2}{\lambda_0^2}\right), \tag{S42}$$

$$U_{,22}(0,0) = 8k\left(\Big(\frac{b^2}{d}(c-d) - \frac{a^2b^2}{\lambda_0^2}\Big)\frac{\lambda_0 - \overline{\lambda}}{\lambda_0} + \frac{a^2b^2}{\lambda_0^2}\right), \tag{S43}$$

$$U_{,12}(0,0) = 8k\Big(\frac{a^2b^2}{\lambda_0^2}\frac{\lambda_0 - \overline{\lambda}}{\lambda_0} - \frac{a^2b^2}{\lambda_0^2}\Big). \tag{S44}$$

The Hessian of the energy, computed in $(\theta_1, \theta_2) = (0,0)$, can be written as

$$\partial^2_{\mathbf{p}} U = \mathbf{K}_T = \mathbf{K}_M + \mathbf{K}_G, \tag{S45}$$

with

$$[\mathbf{K}_G] = 8k\frac{\lambda_0 - \overline{\lambda}}{\lambda_0}\begin{bmatrix} -\frac{a^2}{c}(c-d) - \frac{a^2b^2}{\lambda_0^2} & \frac{a^2b^2}{\lambda_0^2} \\ \frac{a^2b^2}{\lambda_0^2} & \frac{b^2}{d}(c-d) - \frac{a^2b^2}{\lambda_0^2} \end{bmatrix},$$

and

$$[\mathbf{K}_M] = 8k\frac{a^2b^2}{\lambda_0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Internal mechanisms consistent with the D_2 symmetry have the form

$$[\Delta\theta] = \bar{\theta}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{S46}$$

with $\bar{\theta}$ an arbitrary scalar.

The prestress stability condition,

$$\mathbf{K}_G \Delta\theta \cdot \Delta\theta > 0, \tag{S47}$$

gives

$$8\,k\,\varepsilon_0 (c-d)(-\frac{a^2}{c} + \frac{b^2}{d}) > 0, \tag{S48}$$

where $\varepsilon_0 = (\lambda_0 - \overline{\lambda})/\lambda_0$. Since $c > d$, we have

$$\frac{b^2}{d} > \frac{a^2}{c}, \tag{S49}$$

or, by introducing the dimensionless parameters

$$\delta := \frac{b}{a}, \qquad \gamma := \frac{d}{c}, \tag{S50}$$

we can rewrite the *prestress stability condition* as

$$\gamma < \delta^2\,. \tag{S51}$$

We rewrite the component of the stiffness matrix, up to a multiplicative positive constant, as follows:

$$(\mathbf{K}_T)_{11} = \Big(-\frac{a^2}{c}(c-d) - \frac{a^2 b^2}{\lambda_0^2}\Big)\varepsilon_0 + \frac{a^2 b^2}{\lambda_0^2}, \tag{S52}$$

$$(\mathbf{K}_T)_{22} = \Big(\frac{b^2}{d}(c-d) - \frac{a^2 b^2}{\lambda_0^2}\Big)\varepsilon_0 + \frac{a^2 b^2}{\lambda_0^2}, \tag{S53}$$

$$(\mathbf{K}_T)_{12} = \frac{a^2 b^2}{\lambda_0^2}\varepsilon_0 - \frac{a^2 b^2}{\lambda_0^2}. \tag{S54}$$

By setting

$$A = \frac{a^2 b^2}{\lambda_0^2}(1-\varepsilon_0)\,, \qquad B = -\frac{a^2}{c}(c-d)\,\varepsilon_0\,, \qquad C = \frac{b^2}{d}(c-d)\,\varepsilon_0\,, \tag{S55}$$

we can compute the eigenvalues as the solutions ξ of the sequation

$$\det \begin{bmatrix} A+B-\xi & -A \\ -A & A+C-\xi \end{bmatrix} = 0\,, \tag{S56}$$

obtaining

$$\xi^2 - (2A+B+C)\xi + AB + AC + BC = 0\,, \tag{S57}$$

$$2\xi = 2A + B + C \pm \sqrt{4A^2 + B^2 + C^2 - 2BC}\,. \tag{S58}$$

By requiring the lowest eigenvalue to be positive, we have

$$(2A+B+C)^2 > 4A^2+B^2+C^2-2BC \quad \Rightarrow \quad BC+A(B+C)>0\,. \tag{S59}$$

By considering that $\varepsilon_0 > 0$, the condition above amounts to requiring that

$$-\varepsilon_0\left(\frac{c-d}{c\,d}+\frac{1}{\lambda_0^2}\left(\frac{b^2}{d}-\frac{a^2}{c}\right)\right)+\frac{1}{\lambda_0^2}\left(\frac{b^2}{d}-\frac{a^2}{c}\right)>0\,, \tag{S60}$$

or,

$$\varepsilon_0 < \frac{1}{1+\dfrac{1-\gamma}{1-\dfrac{\gamma}{\delta^2}}\dfrac{1}{\sin^2\alpha}} =: \varepsilon_{\mathbf{crit}}\,, \tag{S61}$$

where $\sin\alpha = b/\lambda_0$, with $\alpha = \frac{1}{2}\widehat{EAF}$.

III. POLIGONAL-BASE UNITS

The present calculations can be extended to analogous tensegrity units with polygonal base, such as those shown in Fig S3.

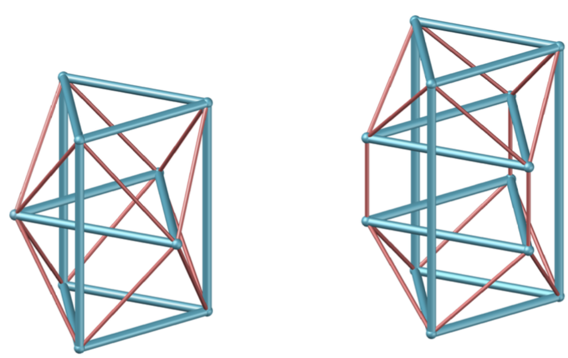

FIG. S3. Units with triangular base.