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Understanding tensegrity with an energy function

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ABSTRACT: The use of a simple quadratic 'energy function' is explored to show how it can be useful in understanding the behaviour of structures that are stressed. The energy function is quadratic in the coordinates of the structure, and is able to correctly capture the stiffness of inextensional modes of deformation that are found in, for instance, tensegrity structures. The paper shows how the quadratic energy function can be used to derive a 'stress matrix' that is useful both for form-finding, and for checking stability.

1 INTRODUCTION

In this paper, we explore how a simple 'energy function' that is a quadratic in nodal coordinates can be useful in understanding the behaviour of structures that are stressed. In particular, this simple energy function Q can capture the additional stiffness that a pin-jointed truss structure has due to its prestress. In most circumstances, this additional stiffness is of negligible order compared with the stiffness associated with the stretching of members, which is the stiffness that is captured by the traditional assumptions of structural mechanics. However, some structures have modes of deformation where, to a first-order approximation, there is no extension of any member — infinitesimal mechanisms. Traditional structural mechanics analysis assigns such modes a zero stiffness: for these modes, it is essential to understand if the stiffness of the mode is positive, so that the mode is stable, or the stiffness is negative, and the mode is unstable. The quadratic energy function allows us to develop simple methods to do this.

The quadratic energy function is particularly useful in understanding and designing tensegrity structures. There is some debate on exactly what constitutes a 'tensegrity' structure, but commonly a tensegrity will be a pin-jointed truss structure that can be self-stressed, so that each member is either a 'cable' that carries tension, or a 'strut' that carries compression (Connelly & Guest 2022). Many tensegrities have struts that are isolated from one another in a net of cables. These are usually the most visually striking examples of these structures: many were built by the sculptor Kenneth Snelson (Heartney 2009), who is widely considered to be the inventor of tensegrity (Snelson 1996).

Tensegrities are commonly underbraced structures (Calladine 1978), and hence it is essential to have methods for form-finding and analysis that correctly calculate the stiffness of the inextensional modes of deformation that are present in these structures.

Two examples of tensegrity structures are shown in Figure 1.

2 ENERGY FUNCTIONS FOR A SINGLE TENSION MEMBER

We begin by considering the very simple two-dimensional system shown in Figure 2. We make two assumptions about the behaviour of the bar that should seem entirely reasonable, at least for small deformations. Firstly, we assume that the bar is *elastic*, so that after any amount of deformation, it will revert to a rest-length (of l_0) after the loading is removed. Secondly, we assume that the bar has a constant stiffness g, so that if the bar has a current length of l, it carries an tension $t = g(l - l_0)$ (of course, this internal force could also have a negative value, when the bar is compressed). For an actual bar with cross-sectional area A, made of a material with Young's Modulus E, this stiffness for small deformations would be $g = AE/l_0$.

2.1 *An 'exact' energy function*

The exact energy function E for the energy stored in the bar is given by

$$E = \frac{1}{2}g(l - l_0)^2, \tag{1}$$

where

$$l = (x^2 + y^2)^{\frac{1}{2}}. \tag{2}$$

We note the straightforward calculation that the force carried is given by

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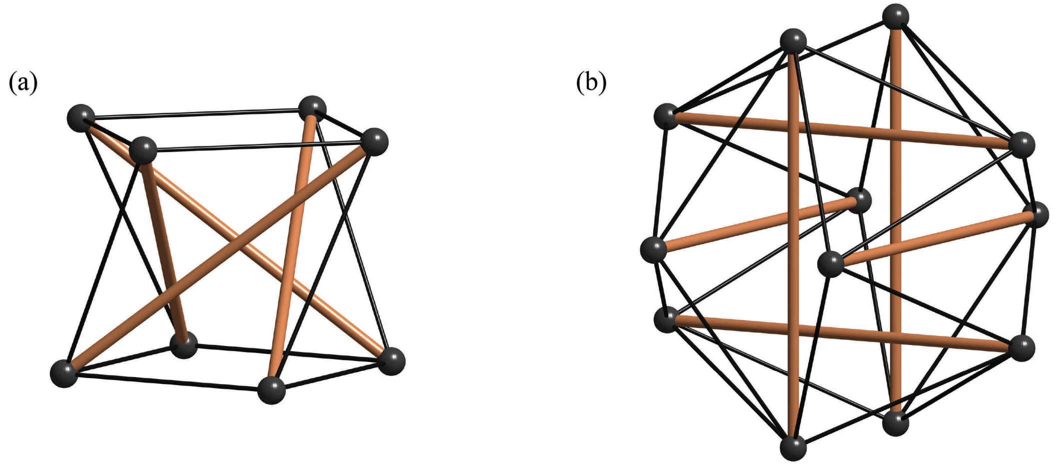

Figure 1. Two examples of a tensegrity structure. In each case, the 'struts' that carry compression are shown by copper-coloured member, while the 'cables' that carry tension are shown by thinner members. The structural connections at nodes can transmit no moment ('pin'-jointed, or perhaps better, 'spherically'-jointed), and are shown as spheres at the ends of members. In both cases, none of the struts touch one another. (a) A 'T4' tensegrity (Pizzigoni et al. 2019), consisting of 8 nodes and 16 members: it has three infinitesimal mechanisms and one state of self-stress. (b) An 'icosahedral' tensegrity (Guest 2011), consisting of 12 nodes and 30 members: it has a single infinitesimal mechanisms and a single state of self-stress.

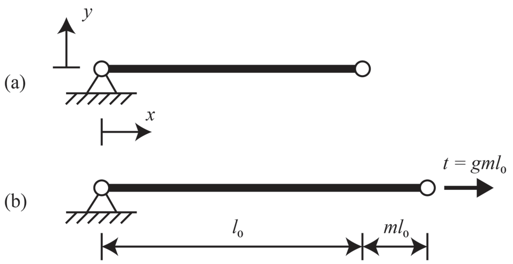

Figure 2. A single bar in two dimensions. One end is constrained at location $(0,0)$, while the other is at point (x,y). (a) The initial unstressed configuration is assumed to lie along the x-axis, so that $(x,y) = (l_0,0)$. (b) We also consider initially stressed configurations lying along the x-axis. Here, the bar is extended by ml_0, which requires a tension $t = gml_0$.

$$t = \frac{dE}{dl} = g(l - l_0), \tag{3}$$

The energy function E is, of course, circularly symmetic about the origin. The function is plotted as a surface in Figure 3(a).

2.2 *The 'linearized geometry' energy function*

The usual approach to analyzing truss structures linearizes the expression for the extension of the bar, $l - l_0$, around the original configuration. For our single bar, with an initial configuration $(x,y) = (l_0,0)$, the variation of length with y, $dl/dy = 0$, and so the y-coordinate is ignored. This means that the energy function L that is assumed is given by

$$L = \frac{1}{2}g(x - x_0)^2. \tag{4}$$

The components of the force applied at the free node given by

$$f_x = \frac{\partial E}{\partial x} = g(l - l_0)\frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \tag{5}$$

$$f_y = \frac{\partial E}{\partial y} = 0. \tag{6}$$

The energy function L is plotted as a surface in Figure 3(b).

2.3 *The 'quadratic' energy function*

Consider an energy function

$$Q_m = \frac{1}{2}g\left(\frac{m}{1+m}\right)l^2 - \frac{1}{2}gml_0^2. \tag{7}$$

This function has been chosen so that both the energy, and the rate of change of energy with length (and hence the tension), is the same for Q_m and the exact function E, at the configuration where $l = (1+m)l_0$.

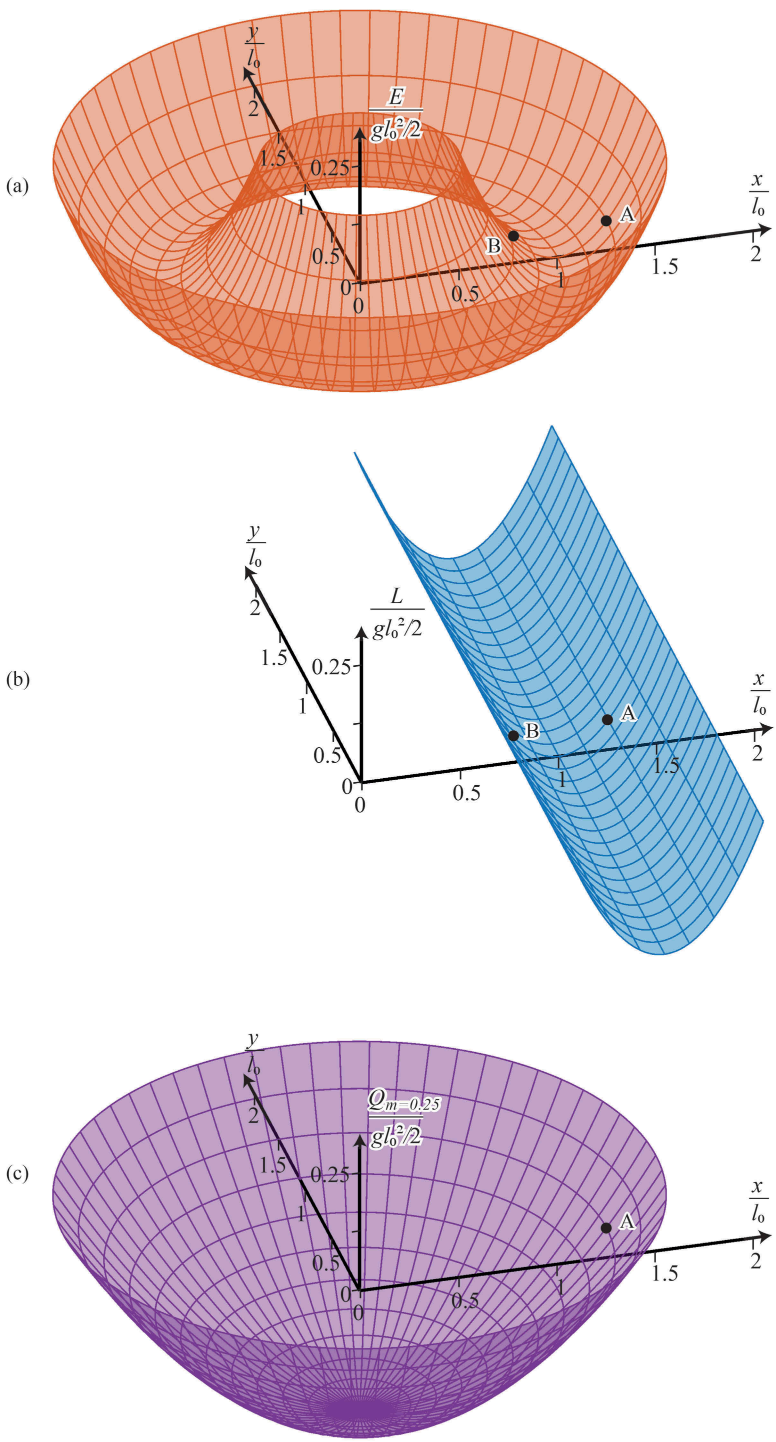

Figure 3. Energy functions for a single bar, non-dimensionalized by a standard energy value $\frac{1}{2}gl_0^2$, plotted varying with two non-dimensional coordinates x/l_0 and y/l_0. Values of energy greater that $0.25 \times \frac{1}{2}gl_0^2$ have been removed for clarity. Three energy functions are shown: (a) the 'exact' energy function E; (b) the 'linearized geometry' energy function L; (c) the 'quadratic' energy function Q_m for $m = 0.25$. All three functions have the same value and the same slope at the point A marked, which is where the bar has the configuration shown in Figure 2b with $m = 0.25$. Point B is marked for the first two plots, where $m = -0.25$.

$$Q_m|_{l=(1+m)l_0} = \frac{1}{2}g(ml_0)^2 = E|_{l=(1+m)l_0} \qquad (8)$$

$$\left.\frac{dQ_m}{dl}\right|_{l=(1+m)l_0} = gml_0 = t|_{l=(1+m)l_0} \qquad (9)$$

The energy function Q_m, with $m = 0.25$, is plotted as a surface in Figure 3(c). This energy plot might at first sight appear to be rather unphysical, corresponding as it does to a bar with an initial rest-length of zero. However, in fact such zero-length springs are an important component of "zero-stiffness" structures, and "statically-balanced" mechanisms (Schenk et al. 2007, Schenk and Guest 2014). By contrast, the energy function Q_m with m negative *is*, by itself, completely unphysical, as it corresponds to an unstable system with an infinite rest length. Nonetheless, as we shall see, it is useful because it captures the instability of a compressed bar to lateral movement of a node.

2.4 *Comparison of energy functions*

Here we explore the three energy functions around the initial configuration of the bar. In Figure 4 we show cross-sections through the function for an initial configuration A, where the bar is extended with $m = 0.25$, and in Figure 5 around a configuration B, where the bar is compressed, with $m = -0.25$. In each case, the value and the slope of the energy functions match. But what is of interest here is to examine the *stiffness* of the system, i.e., the curvature of the functions.

For displacements of the bar in the axial direction, the standard methods of structural analysis inherent in L correctly gives the stiffness — indeed, the function is an exact match to the 'exact' stress function E. And for a configuration where $m = 0$, i.e., the bar is unstressed, the stiffness in the lateral direction would also match, and be equal to zero (this case is not plotted). But when the bars are stressed in tension or compression, the lateral stiffness changes, and this is not captured by L, which continues to predict a zero stiffness.

In fact, for lateral movement, it is the quadratic stress function Q that matches the curvature, and hence correctly predicts the stiffness of the stressed bars. Here it shows that the configuration A is stable, whereas the configuration B is unstable. Thus, for considering the stability of tensegrity structures, when we have to rely on stiffness that does not come from the axial extension of bars, the 'quadratic' stress function can be very useful.

It is a reasonable criticism of the quadratic stress function that it gets the axial stiffness wrong, and indeed for compressive system is physically unrealistic. But in fact it is straightforward to add this stiffness back through a conventional linearized geometry stress function centred on the current configuration (Guest 2006).

3 DERIVATION OF THE STRESS MATRIX

Until now, we have been considering a single bar in isolation. But for a structure with a number of bars that are connected, such as the braced square shown in Figure 6, we need to consider the summation of the stress functions in each bar of the bars. We now also consider the straightforward extension of the earlier material to three dimensions.

3.1 *Single bar*

Consider the single bar $\{1,2\}$ shown in Figure 7. The bar has length $l = (1+m)l_0$, and carries a tension $t = gml_0$. The quadratic stress function for this bar we write as $Q_{\{1,2\}}$, where

$$Q_{\{1,2\}} = \frac{1}{2}g\left(\frac{m}{1+m}\right)l^2. \qquad (10)$$

Compared with the earlier formulation in (7), we have removed the constant term, as the choice of datum is arbitrary for energy, because we are only interested in derivatives of $Q_{\{1,2\}}$, and because this helps simplify our presentation.

Introducing $t' = t/l$ as the *force density* for the bar, and substituting $t' = t/l = gm/(1+m)$, and $l^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ into (10) allows $Q_{\{1,2\}}$ to be written as

$$\begin{aligned} Q_{\{1,2\}} = &\frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} t' & -t' \\ -t' & t' \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \\ &\frac{1}{2}\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} t' & -t' \\ -t' & t' \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \\ &\frac{1}{2}\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} t' & -t' \\ -t' & t' \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned} \qquad (11)$$

We label the matrix in (11) as the *stress matrix* $\mathbf{S}_{\{1,2\}}$, using the notation common in the mathematical rigidity theory literature (Connelly and Guest 2022) (but noting that this is identical to the force density matrix introduced for form-finding by Schek (1974)).

$$\mathbf{S}_{\{1,2\}} = \begin{bmatrix} t' & -t' \\ -t' & t' \end{bmatrix}. \qquad (12)$$

3.2 *Entire structure*

We now consider a structure made up of a number of nodes, such as the braced squares shown in Figure 6. If we have a number of bars in a set B, then we can write the quadratic stress function for the entire bar as the summation of stress functions for each of the bars in B

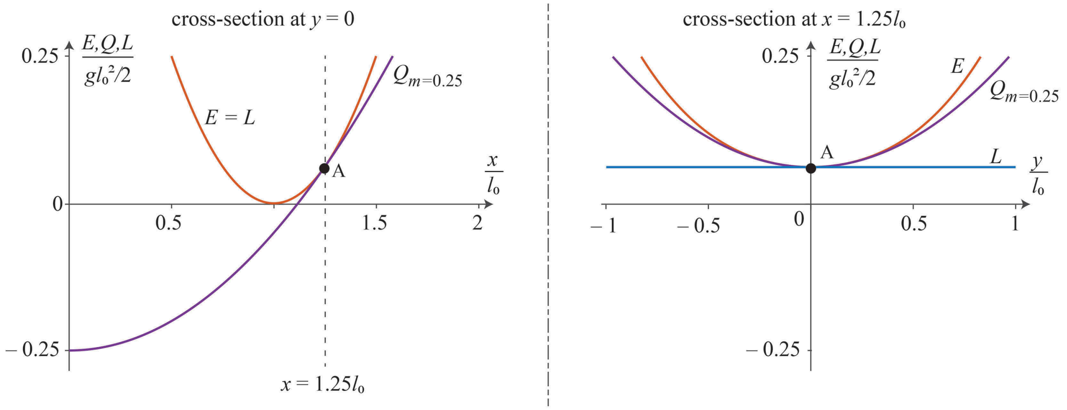

Figure 4. Cross-sections through the energy function, on the left for the plane $y = 0$, and on the right for the plane $x = 1.25l_0$. The energy functions E, L and Q_m for $m = 0.25$ are plotted. The values of the functions, and the slopes, match at the point A, which corresponds to a bar with $m = 0.25$, corresponding to an extension $l - l_0 = 0.25l_0$. For displacement of the end node in the x-direction, the curvatures of the functions E and L (the stiffness) match at A, while for displacement of the end node in the y-direction, it is the curvatures of the functions E and Q_m that match at A.

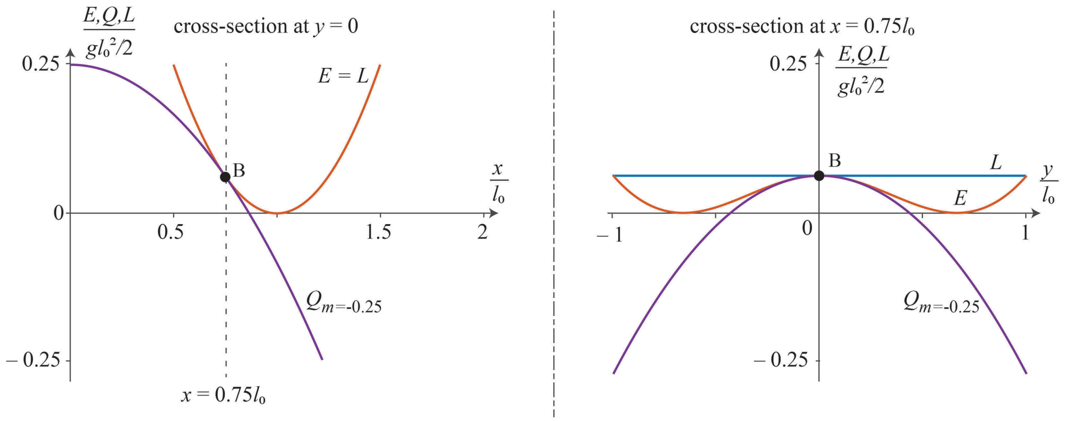

Figure 5. Cross-sections through the energy function, on the left for the plane $y = 0$, and on the right for the plane $x = 0.75l_0$. The energy functions E, L and Q_m for $m = -0.25$ are plotted. The values of the functions, and the slopes, match at the point B, which corresponds to a bar with $m = -0.25$, corresponding to a compression $l_0 - l = 0.25l_0$. For displacement of the end node in the x-direction, the curvatures of the functions E and L (the stiffness) match at B, while for displacement of the end node in the y-direction, it is the curvatures of the functions E and Q_m that match at B.

Figure 6. A planar tensegrity structure, a braced square. Two variants are shown. The first (a) has internal struts carrying compression, the second (b) has internal cables carrying tension. The two crossing members are not joined, and are assumed not to interact. When stressed, (a) forms a stable two-dimensional structure, while (b) is unstable against deformations out of the plane.

$$Q = \sum_{\{i,j\} \in B} Q_{\{i,j\}} \qquad (13)$$

with we can write in terms of the coordinates of the n nodes written as vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix};\ \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix};\ \mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \qquad (14)$$

to give

$$Q = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x} + \frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{S}\mathbf{y} + \frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{S}\mathbf{z} \qquad (15)$$

where **S** is the stress matrix **S** for the entire structure.

The stress matrix in (15) can be assembled from the submatrices for each bar such as $\mathbf{S}_{\{1,2\}}$ given in (12). Alternatively and equivalently, **S** can be assembled directly using the following rules.

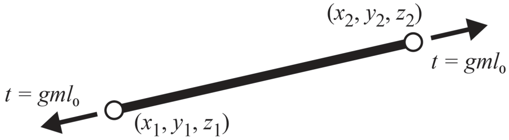

Figure 7. A single bar joining node 1 and node 2, with a length $l = (1+m)l_0$, carrying a tension $t = gml_0$. The force density $t' = t/l = gm/(1+m)$.

- For an off-diagonal term (i,j), the entry is the negative of the force density in the rod connecting i and j if a member exists, or zero otherwise, i.e., $S_{\{i,j\}} = -t'_{\{i,j\}}$ if $\{i,j\} \in B$, or $S_{\{i,j\}} = 0$ otherwise.
- For a diagonal term (i,i), the entry is the sum of the force density in each rod connected to node i, i.e., $S_{\{i,i\}} = \sum_{\{i,j\} \in B} t'_{\{i,j\}}$.

A couple of consequences of the construction are that **S** is a symmetric matrix, and that the entries in each column, or each row, sum to zero — guaranteeing that the all-ones vector lies in the nullspace of **S**.

4 USE OF THE STRESS MATRIX

The stress matrix derived from the quadratic stress function can do double duty, both for equilibrium as a form-finding tool, and to understand the stiffness of a stressed structure.

For equilibrium, consider writing the forces at each node in the x-direction as $\mathbf{f}_x$. For a self-stressed structure, $\mathbf{f}_x = \mathbf{0}$, and so we can write

$$\mathbf{f}_x = \frac{\partial Q}{\partial \mathbf{x}} = \mathbf{S}\mathbf{x} = \mathbf{0} \qquad (16)$$

with similar equations in the y and z directions. The solutions to these equations must be orthogonal to the all-ones solution that we know lies in the nullspace of **S** from its construction. Thus, if we want to have a d-dimensional structure, S must have a nullity of $d+1$ to allow $d+1$ independent solutions to the equation $\mathbf{S}\mathbf{x} = \mathbf{0}$.

For stability, consider that the energy Q should not decrease for any possible displacement of the nodes. Thus **S** must be positive semi-definite. The way in which **S** contributes to the overall stiffness of a prestressed structure is explored more fully in Guest (2006).

If the stress matrix has nullity $d+1$ and is positive semi-definite, it can be termed *super stable*, and it is super stable structures that are the key to finding robust tensegrities of the type shown in Figure 1. More details can be found in Connelly and Guest (2022).

4.1 *Example: A braced square*

For the braced square shown in Figure 6, the connectivity of the structure defines the stress matrix **S** to be,

$$\begin{bmatrix} t'_{\{1,2\}}+t'_{\{1,3\}}+t'_{\{1,4\}} & -t'_{\{1,2\}} & -t'_{\{1,3\}} & -t'_{\{1,4\}} \\ -t'_{\{1,2\}} & t'_{\{1,2\}}+t'_{\{2,3\}}+t'_{\{2,4\}} & -t'_{\{2,3\}} & -t'_{\{2,4\}} \\ -t'_{\{1,3\}} & -t'_{\{2,3\}} & t'_{\{1,3\}}+t'_{\{2,3\}}+t'_{\{3,4\}} & -t'_{\{3,4\}} \\ -t'_{\{1,4\}} & -t'_{\{2,4\}} & -t'_{\{3,4\}} & t'_{\{1,4\}}+t'_{\{2,4\}}+t'_{\{3,4\}} \end{bmatrix}$$

To give a two-dimensional structure, **S** must have a nullity of 3, and hence a rank of 1, i.e., each row must be the same apart from a scaling. Also, each row must be orthogonal to the all ones vector. Thus we get a unique solution (up to relabelling of coordinates) where the magnitude of the force density is equal to t' in every bar, but has opposite sign in the diagonals compared to the other bars.

$$\mathbf{S} = \begin{bmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{bmatrix} t' \qquad (17)$$

For the case where t' is positive, cables form the outer members of the square, **S** is positive semi-definite and the structure is stable (Figure 6(a)). For the case where t' is negative and cables form the diagonal members **S** is negative semi-definite and the structure is unstable (Figure 6(b)) — this stressed structure is stable in the plane, because the conventional structural action adds stiffness in the plane, but the structure is unstable, and will collapse, out of the plane.

5 CONCLUSION

A simple quadratic stress function has been shown to be useful to understand the behaviour of underbraced structures, and in particular whether they are stable in an unbraced mode.

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