MATRICES IN THE TEACHING OF STATICALLY INDETERMINATE STRUCTURES

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An experienced structural engineer can usually calculate the degree of statical indeterminacy of a structure using ad-hoc methods, and can select suitable "cuts" to make the structure statically determinate. Students, however, often find this an altogether puzzling business. Here we describe a technique that is taught in our Department for analysing statically indeterminate structures by use of simple equilibrium matrices; straightforward manipulation of these matrices reveals the number of statical indeterminacies, and any states of self-stress — or indeed mechanisms — that may exist in a given structure.

INTRODUCTION

The theory of structures is an essential part of any university course in Civil or Structural Engineering.

In most institutions the teaching of engineering subjects is done "from the particular to the general". By this we mean that teaching is done in such a way that important ideas on any subject are introduced to the students in the first instance through relatively simple examples. Then, once the key ideas have been grasped, more detailed ramifications can be developed.

Thus, in teaching the theory of structures it is usual to begin with *statically determinate* structures. For this type of structure, of course, the "internal forces" can be related to the "external loads" by considerations of statical equilibrium alone. The key idea in this connection is the "free-body diagram",

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which enables internal forces to be accessed by consideration of the equilibrium of a suitable piece of the structure that has been isolated by cuts through members across which the internal forces act. The student is rapidly brought face-to-face with the important idea that the engineer needs to know what forces are being transmitted through the various members of the structure when a given set of loads is applied to it. Later on, there can be discussion of the elastic response of these members to the internal forces and stresses which they carry; and then the geometrical business of determining the distortion of the structure as a whole on account of the known deformations of the individual members can be tackled. So the statically determinate structure — whether a triangulated framework, a beam or a pressure vessel — provides a way of showing the student various kinds of problem which structural engineers face, and how they can be solved by invoking conditions of equilibrium, the constitutive relations for the material, and the kinematics of distortion.

But most real structures are actually statically indeterminate; and so the analysis of such structures cannot be postponed for long. Probably the simplest example of a statically indeterminate structure for a student to grasp is the "propped cantilever". This provides an introduction to several important ideas. Thus, the structure can sustain a state of *self-stress*, that is a state of stress within the structure when there is zero external loading. Also, it is possible through this example to introduce the important idea that the state of stress within a structure under load cannot be fully determined without a knowledge of the initial state of self-stress in the structure. In some textbooks this particular point is not emphasised; and problems can, of course, be posed which sidestep this issue by the use of words such as "determine the stresses in the structure on account of the application such-and-such a load".

The most straightforward way of introducing a systematic analysis of statically indeterminate structures is in the context of triangulated frameworks, composed of straight members connected to each other at their ends by frictionless "pin" joints. Such structures will have been introduced, of course, in the context of statically determinate structures; but they can be used to illustrate statically indeterminate structures by having a few extra members inserted between the existing joints.

It is at this point in many courses that an air of mystery is introduced into the proceedings. Thus the student may be asked to consider a number of examples of two-dimensional frameworks of this kind, and to state, by inspection of each case, how many degrees of statical indeterminacy there are; or how many "redundant" bars there are; or how many "releases" of bars or supports are needed to render the structure statically determinate. Some students, who have a good "structural imagination" may find this rather straightforward; but many students tend to find it an altogether puzzling business.

In this paper we shall describe a scheme for teaching statically indeterminate structures by means of matrices — a scheme which has been used successfully in our Department for several years. Static and kinematic indeterminacy in elastic frameworks is described completely by *linear algebra*. But in order to make progress in understanding, we must break away from the not uncommon notion that the only useful matrices are *square*, and get to grips with situations where the number of equations differs from the number of unknowns.

For the sake of brevity we shall here consider only a few simple two-dimensional frameworks; but the same methods are of course more generally applicable. First we shall describe the examples in physical terms, and then we shall deploy the mathematics of linear algebra.

Figure 1(a) shows a framework of four bars ($b = 4$) and two pin joints ($j = 2$) — not counting the foundation joints. We say that it is *statically determinate* if the forces in all bars can be determined uniquely for any given loads (i.e. forces, applied to the joints) by solving the equations of equilibrium. Here there are two (component) equilibrium equations for each joint, so the number of equilibrium equations is equal to the number of unknown bar forces; and indeed the assembly is statically determinate. The condition

$$b = 2j, \qquad (1)$$

here satisfied, is known as "Maxwell's rule" for the assembly to be statically determinate (Calladine, 1978). Actually, Maxwell was mainly concerned with the conditions for the frame to be *rigid*; i.e. for the joints to be immovable if the bars were of fixed length. That is a problem of kinematics. But the same rule applies, since the two joints, if detached, would have together four degrees of freedom, while the rigid bars provide four constraints.

Figure 1(b) shows the same structure, to which has now been added a second diagonal bar. Now

$$b > 2j, \tag{2}$$

and we say that the assembly is *statically indeterminate*: there are now more force variables than the number of equilibrium equations, and so the forces cannot be uniquely determined by equilibrium considerations alone. The extra bar will only fit, of course, if it is of precisely the right length. Otherwise it will have to be extended or compressed in order to fit; and that will set up a *state of self stress* throughout the structure, i.e. a set of bar forces which is not zero when the external load components are all zero.

Figure 1(c) shows the assembly of Fig. 1(a), but now with the diagonal removed. Here,

$$b < 2j \tag{3}$$

and the assembly is a *mechanism*, with one degree of freedom: the square linkage can now freely deform into a rhombus. We say that it is *kinematically indeterminate* in the sense that the joints are not uniquely located by the (inextensional) bars. The arrangement can sustain external loads, but only if these do not "excite" the mechanism. For the present example, external loads can be carried (in the "square" conformation) only if $P_1 + P_3 = 0$.

Figure 2(a) is an example which satisfies Maxwell's rule (1), with $b = 8$ and $j = 4$; but it is clearly *both* statically indeterminate (since the lower part can sustain a state of self-stress, as in Fig. 1(b)) *and* kinematically indeterminate (since the upper part has a degree of kinematic indeterminacy, just as in Fig. 1(c)).

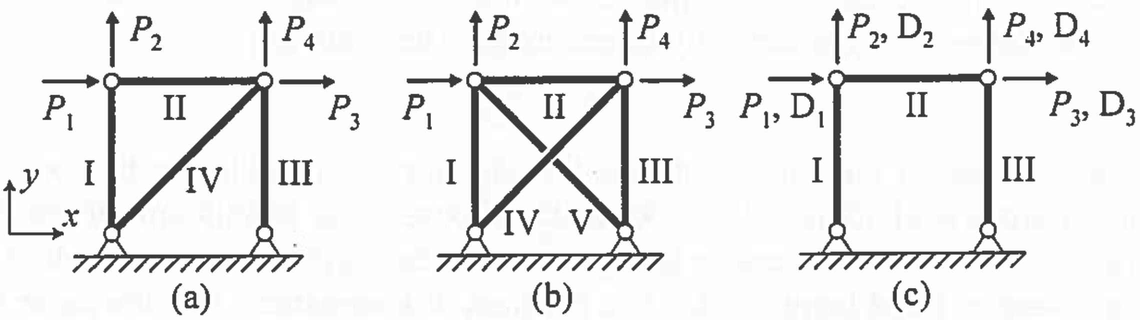


Figure 1. Three pin-jointed frameworks. $P_1 \cdots P_4$ are components of the loads applied at the two non-foundation joints, and $D_1 \cdots D_4$ are the corresponding components of deflection. The bars are numbered by roman numerals.

Lastly, Fig. 2(b) is an example which also satisfies (1), with $b = 8$ and $j = 4$. But is it consequently statically and kinematically determinate, as in the assembly of Fig. 1(a)? The answer is *no*; the assembly can sustain a state of self-stress *and* it is also a mechanism having a single degree of freedom (but for small displacements/ rotations only).

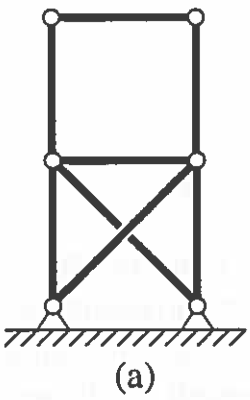
It is thus clear from Figs 2(a) and (b) that the Maxwell rule (1) is not a *guarantee* of static and kinematic determinacy. But while the example of Fig. 2(a) seems readily comprehensible as a sort of self-compensating combination of the examples of Figs 1(b) and (c), the example of Fig. 2(b) is evidently more subtle: the assembly as a whole is in fact both statically and kinematically indeterminate.

Examples such as Fig. 2(b) have puzzled engineers and others for a long time, and numerous attempts have been made over the years to work out the circumstances in which an assembly can violate Maxwell's rule in this way. The key to the situation, as we now know, is to apply the methods of linear algebra to the equilibrium equations, and to let the structural characteristics of particular assemblies emerge from the mathematical working.

CALCULATIONS

Example 1(a)

Our first task is to set up the equations of equilibrium for the assembly of



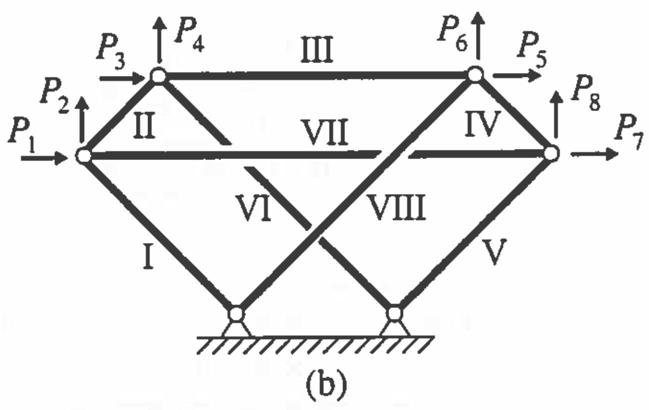


Figure 2. Two pin-jointed frameworks that are simultaneously statically and kinematically indeterminate

Fig. 1(a). The external load components P_1 to P_4 are applied to the two joints; P_1 and P_2 to the first joint in the x- and y- directions, and likewise P_3 and P_4 to the second joint. The bars are numbered I to IV, in an arbitrary fashion, as shown. Each bar i carries a force T_i, which is positive when tensile.

The equations of statical equilibrium can now be written down. Resolving forces horizontally and vertically at the joints in turn, and putting the equations into matrix form, we find (4a); here, 0.7 stands for $1/\sqrt{2}$.

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.7 \\ 0 & 0 & 1 & 0.7 \end{bmatrix} \begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \\ T_{\mathrm{IV}} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \tag{4a}$$

We are interested in the solution(s) of these equations. Following Strang (1988), we eschew the use of computer software at this stage, and make a direct *Gauss-Jordan elimination*. The first stage of this process is to perform row manipulations (equivalent to making linear combinations of the original equations) in order to re-arrange the matrix in *echelon form*, i.e. a matrix with 1 as each element of the leading diagonal and zeros in all lower elements. In the present case we can achieve most of this objective by putting the first equation last and moving each of the others up one row. The fourth equation is found by adding the first and third equations of (4a) and then multiplying by $\sqrt{2}$. A steadily-descending "staircase" has been put in below each of the "pivots". The equations are sufficiently simple in this case to be handled with the external forces expressed symbolically.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.7 \\ 0 & 0 & 1 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \\ T_{\mathrm{IV}} \end{bmatrix} = \begin{bmatrix} P_2 \\ P_3 \\ P_4 \\ 1.4(P_1 + P_3) \end{bmatrix} \tag{4b}$$

Inspection of (4b) reveals that we can proceed to obtain a unique value for each of T_{I} to T_{IV} for given P_1 to P_4 by starting with the lowest equation and working our way upwards. The echelon form guarantees that there is no ambiguity in the outcome at any stage of the process. The final result is shown in (4c), which may readily be checked by the reader. The matrix has been *diagonalised*: we now have a set of four equations which give, one by one, the forces in the four bars in terms of the given external loads. The solution of the equations is *unique*. In particular, if the external loads are all zero, so also are the bar forces, and hence a state of prestress cannot exist.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \\ T_{\mathrm{IV}} \end{bmatrix} = \begin{bmatrix} P_2 \\ -P_1 \\ -P_1 - P_3 + P_4 \\ 1.4(P_1 + P_3) \end{bmatrix} \tag{4c}$$

Example 1(b)

Turning to the assembly of Fig. 1(b), we have set up the corresponding equilibrium matrix in (5a). Here there are five unknown bar forces, and so the matrix now has five columns; but there are still only four equations of equilibrium, and hence four rows. Note that the first four columns are unchanged from (4a).

$$\begin{bmatrix} 0 & -1 & 0 & 0 & -0.7 \\ 1 & 0 & 0 & 0 & 0.7 \\ 0 & 1 & 0 & 0.7 & 0 \\ 0 & 0 & 1 & 0.7 & 0 \end{bmatrix} \begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \\ T_{\mathrm{IV}} \\ T_{\mathrm{V}} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \tag{5a}$$

We now proceed as before to put the matrix into echelon form; and the outcome is shown in (5b). On this occasion it is the last column that lacks a pivot, which corresponds to bar 5 being the "redundant" member — the method automatically designates redundant bars. We next make linear combinations of rows in order to diagonalise the matrix; but although we can do this for the first four columns (the results being exactly the same as before, of course), we end up with various entries in the last column, as shown in (5c).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0.7 \\ 0 & 1 & 0 & 0.7 & 0 \\ 0 & 0 & 1 & 0.7 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \\ T_{\mathrm{IV}} \\ T_{\mathrm{V}} \end{bmatrix} = \begin{bmatrix} P_2 \\ P_3 \\ P_4 \\ 1.4(P_1 + P_3) \end{bmatrix} \tag{5b}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0.7 \\ 0 & 1 & 0 & 0 & 0.7 \\ 0 & 0 & 1 & 0 & 0.7 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \\ T_{\mathrm{IV}} \\ T_{\mathrm{V}} \end{bmatrix} = \begin{bmatrix} P_2 \\ -P_1 \\ -P_1 - P_3 + P_4 \\ 1.4(P_1 + P_3) \end{bmatrix} \tag{5c}$$

Inspection shows that the values of T_{I} to T_{IV} cannot be determined uniquely in terms of P_1 to P_4: we need also to know the value of T_{V}. Bar 5 is "redundant", of course. Now if we set $T_{\mathrm{V}} = 0$, temporarily, we can express the solution vector $\left[T_{\mathrm{I}} \cdots T_{\mathrm{V}}\right]^T$ as the first term on the RHS of (5d). And then if we set $P_1 \cdots P_4 = 0$ (i.e. the external loads all to zero) and put $T_{\mathrm{V}} = -1$ we obtain the second term on the RHS. The multiplier α has been inserted here because the values of T_{I} to T_{IV} in this case are all proportional to T_{V}. For this last step we could, of course, have chosen $T_{\mathrm{V}} = 1$; but by putting $T_{\mathrm{V}} = -1$ we can conveniently transfer the first four elements of this column from the fifth column of the matrix of (5c); and T_{V} provides the missing entry.

$$\begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \\ T_{\mathrm{IV}} \\ T_{\mathrm{V}} \end{bmatrix} = \begin{bmatrix} P_2 \\ -P_1 \\ -P_1 - P_3 + P_4 \\ 1.4(P_1 + P_3) \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0.7 \\ 0.7 \\ 0.7 \\ -1 \\ -1 \end{bmatrix} \tag{5d}$$

Equation (5d) represents formally the complete solution of equations (5a). The first column on the RHS represents an equilibrium solution, when $T_{\mathrm{V}} = 0$ ("when we have cut the redundant bar"); while the second column represents the state of self-stress (when there is zero external load) which the structure can sustain ("when a turnbuckle in member V is tightened"). The situation is closely analogous to the Particular Integral and Complementary Function in the solution of ordinary differential equations: the PI is the solution when the RHS = 0.

In order to determine the value of T_{V} (or, equivalently, α) in a given case we need also to bring into consideration both the equations of geometrical compatibility of the assembly, and also the constitutive relations for the bars, involving elastic modulus, coefficient of expansion, and manufacturing error. That is beyond the scope of the present paper; but it is an entirely straightforward matter provided we adhere to systematic procedures.

Although experienced engineers might regard (5d) as a statement of the obvious, which could have been written down after much more "informal" physical computations "on the back of an envelope", the benefits of using linear algebra can only accrue if the proper formalities are adhered to. This point will be particularly well illustrated by Example 2(b).

Example 1(c)

Consider now the assembly shown in Fig. 1(c). Here there are no diagonals, and the equilibrium matrix (6a) consists of only the first three columns of (4a) or (5a):

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \tag{6a}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \end{bmatrix} = \begin{bmatrix} P_2 \\ P_3 \\ P_4 \\ P_1 + P_3 \end{bmatrix} \tag{6b}$$

Proceeding as before we obtain the echelon form (6b). The fourth row of this matrix consists entirely of zeros: it corresponds to the addition of the first and third equations in (6a), and it has $(P_1 + P_4)$ on the RHS. The first three equations of (6b) tell us that T_{I}, T_{II} and T_{III} are equal respectively to P_2, P_3 and P_4. However, we must also satisfy the fourth equation; and this requires

$$P_1 + P_3 = 0 \tag{7}$$

since the LHS is always zero.

The physical meaning of this result is that *the assembly cannot be in equilibrium unless the total horizontal load is zero.* Physically, of course, the assembly is a mechanism with one degree of freedom.

This example provides the cue for investigating the *kinematic* equations of the problem. We define four components of joint displacement, as shown in Fig. 1(c): they are the work conjugates of the load components. We also define the elongations $E_{\rm I}$, $E_{\rm II}$, $E_{\rm III}$ of the bars, being the work conjugates of the corresponding bar forces. Then, provided the displacements of the joints and the rotations of the bars are sufficiently small, we can write the kinematic relations for the assembly as follows, by inspection:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} E_{\rm I} \\ E_{\rm II} \\ E_{\rm III} \end{bmatrix} \tag{8a}$$

This 3 × 4 *compatibility* matrix is, in fact, simply the transpose of the equilibrium matrix of (6a); and this can easily be shown to always be the case, by reason of the of the theorem of Virtual Work.

Suppose we have been given the bar elongations. Solving the matrix equations using the same techniques as previously (but omitting the intermediate steps) we obtain a general solution for the displacements:

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} -E_{\rm II} \\ E_{\rm I} \\ 0 \\ E_{\rm III} \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \tag{8b}$$

Inspection of Fig. 1(c) reveals that the second term on the RHS of (8b) describes precisely the degree of freedom of the assembly as a small-displacement mechanism.

The examples of Figs 1(b) and (c) thus reveal a close formal connection between the *statics* of Fig. 1(b) and the *kinematics* of Fig. 1(c). In each case there are more unknowns than equations; and in each case the Gauss-Jordan elimination provides automatically the details of the static/kinematic indeterminacy. The counterpart of the static constraint (7) on the loads acting on the assembly of Fig. 1(c) is a kinematic constraint on the bar elongations in the structure of Fig. 1(b) if all the bars are to remain connected. This is left as an exercise for the reader.

Example 2(b)

Let us now go directly to the assembly of Fig. 2(b). First we write the joint-equilibrium equations. When we get to the end of the echelon-form computation, we find that there is no pivot in the eighth column. Neglecting this column, and back-substituting from the bottom up, we obtain (9)

$$\begin{bmatrix} 1 & & & & & & & 1 \\ & 1 & & & & & & 1 \\ & & 1 & & & & & 1.4 \\ & & & 1 & & & & 1 \\ & & & & 1 & & & 1 \\ & & & & & 1 & & -1 \\ & & & & & & 1 & -1.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{\mathrm{I}} \\ T_{\mathrm{II}} \\ T_{\mathrm{III}} \\ T_{\mathrm{IV}} \\ T_{\mathrm{V}} \\ T_{\mathrm{VI}} \\ T_{\mathrm{VII}} \\ T_{\mathrm{VIII}} \end{bmatrix} = \begin{bmatrix} 1.4P_2 + 0.7(P_3 + P_4 + P_5 + P_6) \\ 0.7(P_3 + P_4 + P_5 + P_6) \\ P_5 + P_6 \\ 1.4P_6 \\ 1.4(P_1 + P_2 + P_3 + P_4 + P_5 + P_7) \\ 0.7(-P_3 + P_4 - P_5 - P_6) \\ -P_1 - P_2 - P_3 - P_4 - P_5 \\ P_1 + P_2 + P_3 + P_4 + P_5 - P_6 + P_7 - P_8 \end{bmatrix} \quad (9)$$

For equilibrium to be possible, the external loads must also satisfy the last equation; and by analogy with (7) and Fig. 1(c), it seems clear that this is the condition for a mechanism not to be excited. And also, since column 8 contains no pivot, it is clear that the assembly can sustain a state of self-stress, which is given by the final column of the matrix in (9), with $T_{\mathrm{VIII}} = -1$, just as in equations 5(c) and (d). And a kinematic analysis, as in example 1(c), reveals this assembly's degree of kinematic freedom: again this is left as an exercise for the reader.

DISCUSSION

The example of Fig. 2(b) takes us further than our third-year undergraduate course. That course has a lot of examples, and includes a general method for solving problems where there are multiple redundancies/states of self-stress.

The cases presented here have been chosen mainly to illustrate the proposition that the physical character of a given assembly can be assessed most satisfactorily by means of a Gauss-Jordan eliminations of the equilibrium and compatibility equations, rather than by assembling a *stiffness* or *flexibility* matrix in the first instance.

Some texts give the impression that the best way of dealing with the rectangular matrices which frequently represent the equilibrium and compatibility relations is to combine them into a square stiffness matrix. Indeed, stiffness matrices will generally be used by any computer structural-analysis packages that a student may use. However, there is a grave danger here that the student will believe that the elastic solution produced is the correct engineering solution to a problem, and will lose sight of the assumptions that were made in order to provide this "unique" solution. The use of the equilibrium approach allows students to break out of this "Navier's Straightjacket" (Heyman, 1998) and to understand the different ways that a structure can support a load. Another drawback of a stiffness approach is that it gives little clue of how to deal with kinematically indeterminate structures such as cable-nets and the like. Tomorrow's structural engineers need to be able to work with such structures; and we believe that the inclusion of material of the kind that we have presented here provides a good foundation for further study.

An understanding of equilibrium and compatibility relationships maps directly onto the well-known mathematical area of linear algebra. In this paper we have deliberately avoided giving formal names to concepts such as *rank*, *vector space*, *null space*, etc. which our various examples illustrate well. For further study we recommend Strang (1988).

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