Wrapping the Cube and other Polyhedra

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Abstract

An infinite series of 2-fold, 2-way weavings of the cube, corresponding to 'wrappings', or double covers of the cube, is described with the aid of the 2-parameter Goldberg-Coxeter construction. The strands of all such wrappings correspond to the central circuits of octahedrites (4-regular polyhedral graphs with square and triangular faces), which for the cube necessarily have octahedral symmetry. Removing the symmetry constraint leads to wrappings of other eight-vertex convex polyhedra. Moreover, wrappings of convex polyhedra with fewer vertices can be generated by generalising from octahedrites to i-hedrites, which additionally include digonal faces. When the strands of a wrapping correspond to the central circuits of a 4-regular graph that includes faces of size greater than 4, non-convex 'crinkled' wrappings are generated. The various generalisations have implications for activities as diverse as the construction of woven closed baskets and the manufacture of advanced composite components of complex geometry.

1 Introduction

Carbon-fibre composites are used throughout advanced manufacturing, and figure in, for instance, the latest aerospace components (Toensmeier, 2005). In many applications, tows (bundles) of fibres are used in the form of a weave (Onal and Adanur, 2007); in other applications, tows of fibres are wrapped on a former, using tow placement machines (Rudd et al., 1999). Directly related to these modern technologies is the long-established weaving of baskets, open and closed, a technology common to many cultures and

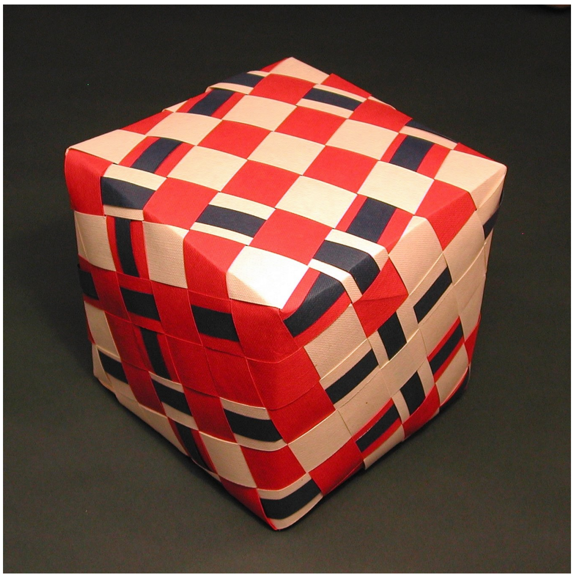


Figure 1: Skew weaving of the cube with $b = 5$ and $c = 2$ (see text for definitions). The dark ribbon is a single closed strand. The whole weaving has 3 symmetrically related strands of length 116 unit squares each. This closed basket was constructed by Felicity Wood of the Oxfordshire Basketmakers Association, who provided the photograph.

periods (Tarnai, 2006; Pitt Rivers Museum, 2009), which continues to generate applications in art and craft (Kavicky, 2004; Pulleyn, 1991) and modern architecture (Ministry of Land, Infrastructure and Transport and Nihon Sekkei Inc., 2005). Closed baskets are often considered as woven spheres or polyhedra, and are treated in many mathematical reviews and books, e.g., Pedersen (1981); Gerdes (1999); von Randow (2004). The present paper examines mathematical aspects of weaving on polyhedral surfaces, with practical applications in mind, concentrating initially on weavings on the cube (Figure 1), before extending the treatment to a class of weavings that turn out to be described by the 'octahedrites' of Deza and Shtogrin (2003).

For the *plane*, the simplest weaving is the 2-fold, 2-way weave (Grünbaum and Shephard, 1988) in which a typical point is covered by two strands (hence '2-fold'), with the strands crossing at right angles (hence '2-way'), in an overall check pattern where an individual strand passes alternately over and under at crossings. The same basic definition can be applied to a weaving of a closed basket on the surface of the cube (Tarnai, 2006), and, as we shall see, to other polyhedra; as on the plane, strands cross at right angles, and each strand passes alternately over and under at crossings. However, the strands are now necessarily of finite length, and may have self-intersections.

For construction and classification, it is convenient to simplify the physical weave to a *double cover*, where the up-down relationship of the strands has been 'flattened out', so that, apart from points on strand boundaries, every point belongs to two orthogonal portions of strands, with no concept of one strand being above another. In what follows, we will be concerned with the properties of this double-cover version of the weaving, which we can informally call a 'wrapping'.

For the particular case of the cube, the strict alternation of the check pattern is necessarily disrupted at vertices, and the symmetries of the overall pattern are restricted to a subset of those of the underlying cube. Weavings of the cube fall into three types, depending on the pairs of angles of intersection between the strand and the cube edges: Class I $(0, \pi/2)$, Class II $(\pi/4, \pi/4)$, Class III $(\theta, \pi/2 - \theta)$, with $0 < \theta < \pi/4$. Examples of all three are illustrated in Figure 2. This classification echoes the schemes for geodesic domes (Coxeter, 1971; Wenninger, 1979), and for classifying carbon nanotubes into armchair, zig-zag and chiral types (Hamada et al., 1992). As wrappings, double covers in Classes I and II have the full set of symmetries of the cube, whereas those in Class III have only the rotations.

Consideration of the ways in which wrappings of the cube can be represented and classified leads naturally to polyhedral graphs, from which it becomes apparent that all cube wrappings can be represented as members of the family of *octahedrite* graphs (Deza and Shtogrin, 2003). Generalisations, by removal of the restriction to octahedral symmetry, by addition of digonal faces to the octahedrite recipe, or by introduction of general face sizes, will be shown to generate further infinite families of convex and non-convex wrapped polyhedra, and hence of closed baskets.

2 Geometrical approach to cube wrappings

A natural representation of a double covering of the *plane* by strands shows the strand boundaries as orthogonal lines. This leads to a tessellation of the plane by square tiles, meeting four at a vertex, i.e., $\{4, 4\}$ in the Schläfli notation (Coxeter, 1969). Each square tile corresponds to two overlaid portions of strands of the original weaving. Many different weavings may correspond to a given double cover (Grünbaum and Shephard, 1988).

Wrappings of the cube can be represented in a similar way by drawing a net of the cube onto the $\{4, 4\}$ tessellation, with the restriction that the vertices of the net are lattice points, i.e., points at which strand boundaries cross. This restriction follows from the observation that a cube vertex cannot lie within a strand, as the sum of angles at a cube vertex is only $3\pi/2$. Each such net can be described by a symbol $\{4, 3+\}_{b,c}$, where the 4 implies a tiling by squares, and the 3+ indicates that three or more square tiles meet at each vertex of the net. The integers b and c ($b \geq 0, c \geq 0, b+c > 0$) define

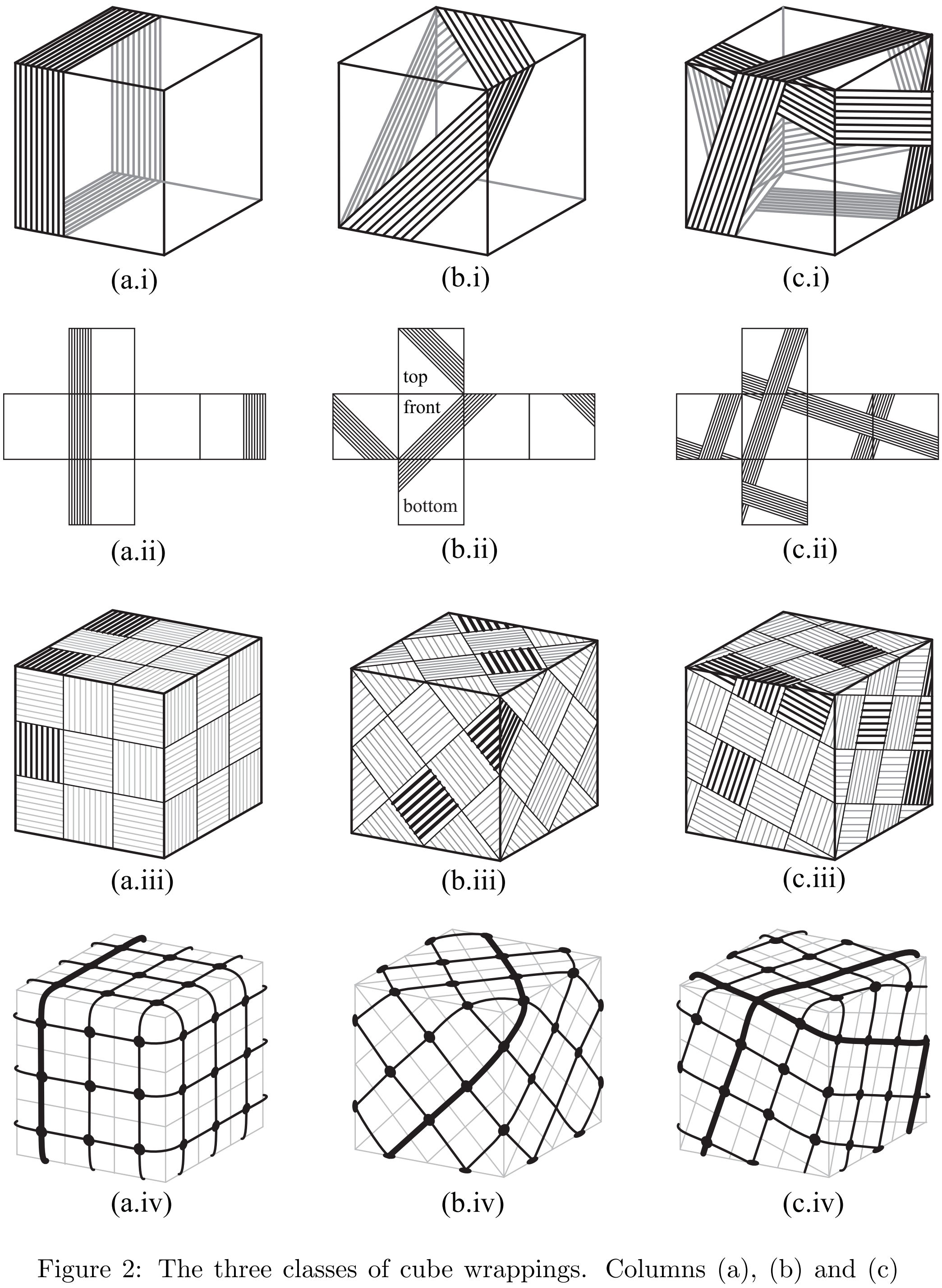


Figure 2: The three classes of cube wrappings. Columns (a), (b) and (c) illustrate wrappings of class I, II and III respectively. The illustrated wrappings are $\{4, 3+\}_{3,0}$, $\{4, 3+\}_{2,2}$ and $\{4, 3+\}_{3,1}$. Row (i) shows a single strand wrapped onto the cube, and row (ii) shows the same strand on the unfolded net of the cube. Row (iii) shows the complete weaving, with the single strand highlighted. Row (iv) shows the dual maps of the wrappings as graphs embedded on the cube.

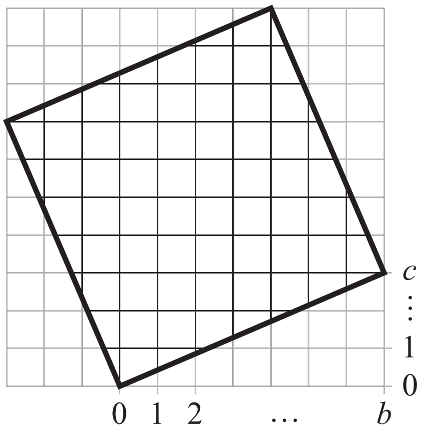


Figure 3: Definition of parameters b and c for tiling $\{4, 3+\}_{b,c}$ of the cube.

how all the (congruent) faces of the net lie on the underlying tessellation of the plane (Figure 3): from any starting vertex, an adjacent vertex is reached by making b steps along edges of the tessellation in one direction, followed by c steps after a change in direction by an angle of $\pi/2$. This 2-parameter construction is ultimately derived from the work of Goldberg (1937) and Coxeter (1971), and has been applied in the present context by several authors (Tarnai, 2006; Dutour and Deza, 2004; Deza and Dutour Sikirić, 2007)

If a square tile has unit area (if each strand has unit width), the area of each face of the net is $S = b^2 + c^2$, and the total length (area) of all strands is therefore $12S$, and the angle at which a strand meets a cube edge is $\tan^{-1}(b/c)$, or its complement $\tan^{-1}(c/b)$. Class I corresponds to the case $b = 0$ or $c = 0$; $\{4, 3+\}_{b,0}$ is identical with $\{4, 3+\}_{0,b}$. Class II corresponds to the case $b = c$. Class III corresponds to $b \neq 0, c \neq 0, b \neq c$, and the pair of Class III wrappings with symbols $\{4, 3+\}_{b,c}$ and $\{4, 3+\}_{c,b}$ are enantiomeric as a consequence of the reflection symmetries of the cube (Dutour and Deza, 2004). Figure 4 shows the tilings of the cube faces for the wrappings $\{4, 3+\}_{b,c}$ for small values of b and c. Given this simple parameterisation, it is easy to explore some basic properties of small examples within the three classes. Numerical experimentation gives the results reported in Tarnai (2006) for the parameter $s(b, c)$, the number of strands in the cube wrapping described by b and c. Note that $s(b, c) = s(c, b)$, as exchange of b and c simply flips the chirality of the wrapping. Further combinatorial information can be obtained from Dutour and Deza (2004) Table 6.

Strand counting for Class I is straightforward. A wrapping in Class I has either $b = 0$ or $c = 0$, and without loss of generality we take $c = 0$. The strands all lie parallel to the cube edges (Figure 2(a)), and the double cover has O_h symmetry; each strand has length $4b$, so $s(b, 0) = 12(b^2 +$

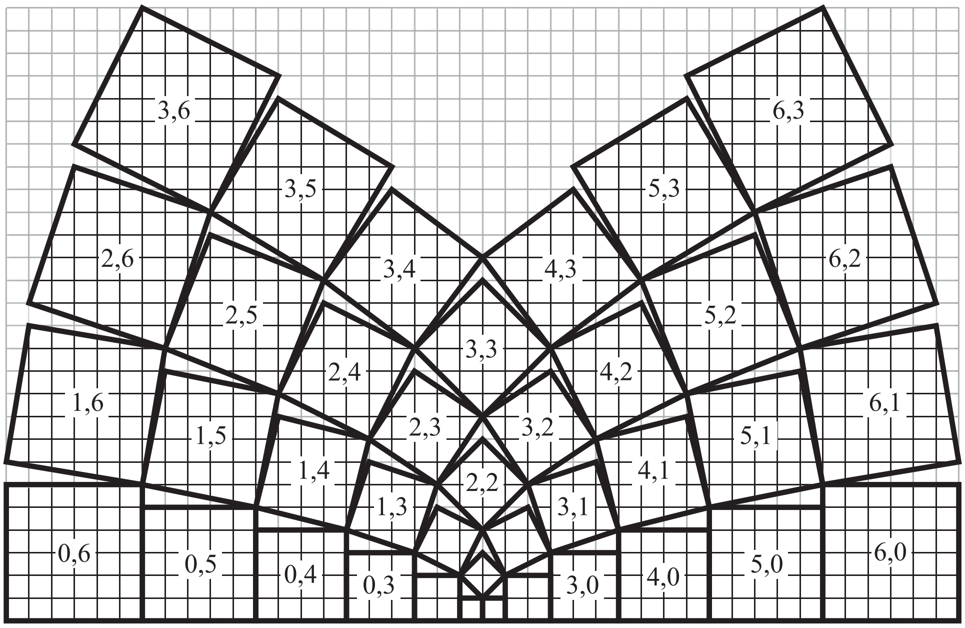


Figure 4: Tilings of the faces of the cube (adapted from Tarnai, 2006), for differing values of the parameter pair b, c. Wrappings belonging to Class I appear (in two copies) along the horizonal axis; wrappings belonging to Class II appear along the central vertical axis; the two enantiomeric versions of each Class III wrapping appear in mirror-symmetric off-axis positions.

$0^2)/4b = 3b$. In Class II, the strands cross the faces at an angle of $\pi/4$ to the edges (Figure 2(b)), and the length of each strand is three times that of the diagonal of a cube face, i.e., $6b$. Hence $s(b,b) = 12(b^2+b^2)/6b = 4b = 4c$.

For counting strands in Class III, another useful observation is that for any pair $b = kb_0$ and $c = kc_0$ with k an integer, the number of strands scales as $s(b,c) = ks(b_0,c_0) = s(c,b)$, and so it is only necessary to understand the behaviour of $s(b,c)$ for the 'canonical' pairs where b and c are co-prime. Where b and c are co-prime, the number of strands is 3, 4 or 6, and the double covering has point-group symmetry D_4, D_3, or D_2, with all strands equivalent.

3 Graph-theory based approach

A more general perspective on the wrapping of cubes (and other polyhedra) comes from a graph theoretical approach. The geometric construction of the previous section specifies a tiling (and so a wrapping) of the cube and hence fixes a polyhedral graph, T, where the faces are the square tiles, the edges are tile edges, and the vertices are tile corners. Note that the vertices of the underlying cube coincide with the 8 three-valent vertices of T, whereas the edges of the cube may run across faces and edges of T. The sum of angles around each 4-valent vertex of T is 2π, and the tiling is 'locally flat' at these points; these vertices all lie on faces or edges of the underlying cube. The cube is the convex realisation of T.

The dual of T, i.e., $T^\star$, is obtained by placing a vertex at the centre of each square tile. The edges of $T^\star$ are then defined by adjacencies (shared edges) of square tiles. Row (iv) of Figure 2 shows the graphs $T^\star$ for three wrappings. As all faces of the primal graph (T) are square, $T^\star$ is a 4-regular graph. The faces of $T^\star$ are either quadrangular, or triangular (corresponding to the eight corners of the cube). This construction suggests the study of 4-regular polyhedral graphs as a basis for systematics of wrappings: $T^\star$ will be a general 4-regular polyhedral graph; its dual will be a tiling T corresponding to a wrapping of an underlying object.

The account below is closely based on the treatment of octahedrite graphs by Deza and co-workers (Deza and Shtogrin, 2003; Deza et al., 2003), and shows how their results can be applied to the wrapping problem for cubes and for more general polyhedra.

3.1 4-regular polyhedral graphs

A polyhedral graph (a graph that is planar and 3-connected (West, 2001)) obeys the Euler theorem

$$v + f = e + 2 \tag{1}$$

where v is the number of vertices, f the number of faces, and e the number of edges. If the graph is 4-regular, $e = 2v$, and if f_r is the number of faces with r sides, we have, by counting faces and edges,

$$\sum_r (4-r) f_r = 8. \tag{2}$$

An immediate consequence is that every 4-regular polyhedral graph has $f_3 \geq 8$ triangular faces. An important subset of 4-regular polyhedra consists of those with the minimum number of triangular faces, and with all other faces quadrangular, i.e., with $f_3 = 8$ and $f_4 = 0$ or $f_4 \geq 2$. These are the polyhedra which are called octahedrites by Deza and Shtogrin (2003), and they are, in a sense, the equivalents amongst 4-regular polyhedra of the fullerenes (Fowler and Manolopoulos, 2006) amongst cubic polyhedra.

The numbers N of octahedrites with n vertices are known for small n (A111361 in Sloane's encyclopedia of integer sequences, (Sloane, 2008; Brinkmann et al., 2003)). They are $N(n)$: 1(6); 0(7); 1(8); 1(9); 2(10); 1(11); 5(12); 2(13); 8(14); 5(15); 12(16); 8(17); 25(18); 13(19); 30(20) .... The point groups allowed for octahedrites comprise the 18 possibilities O_h, O, D_{4d}, D_{3d}, D_{2d}, D_{4h}, D_{3h}, D_{2h}, D_4, D_3, D_2, S_4, C_{2v}, C_{2h}, C_2, C_s, C_i, C_1 (Deza et al., 2003). The subset of octahedrites that have octahedral (O or O_h) symmetry is useful in describing wrappings of the cube; each is the dual, $T^\star$, of a wrapping T.

Some definitions and facts about octahedrites are now briefly summarised. For details and proofs, the original papers by Deza and co-workers (Deza and Shtogrin, 2003; Deza et al., 2003) should be consulted. Graphs of which all vertices are of even degree are *Eulerian*, i.e., they admit circuits that visit every edge exactly once. Eulerian polyhedral graphs have no bridges (cut-edges) and no cut-vertices. For Eulerian graphs embedded in surfaces, we can define *central circuits.* A central circuit (CC) is constructed starting with a single edge, and visiting vertices according to the rule that the sequence enters and leaves any given vertex by opposite edges. For a finite graph, this rule leads to a circuit. The relevance to the wrapping problem is that each CC of the 4-regular graph $T^\star$ corresponds to a strand in the weaving of T (with the CC being the mid-line of the strand).

Central circuits may be *simple*, or *self-intersecting.* The set of CC's partitions the set of edges of the graph. In a 4-regular graph, the full set of CC's provides a double cover of the vertices: every vertex is visited twice by CC's, either once each by two distinct CC's, or twice by a single self-intersecting CC. The length of every CC is even, and the total length of all CC's in the graph is $2n$, by the double-cover property, with n the number of vertices of the 4-regular graph (the number of squares in the primal). Thus, all strands are of even length, and, for a cube wrapping, their total length is $12S = 12(b^2 + c^2)$.

A *railroad* is a circuit of square faces in an octahedrite. Octahedrites without railroads are called *irreducible*. In the context of cube wrappings, the duals of the wrappings with b and c co-prime are irreducible octahedrites. It can be proved that every irreducible octahedrite, of whatever symmetry, has at most six CC's. Thus, for cube wrappings with b, c co-prime, the number of strands is limited to 3, 4, or 6, as noted earlier. There are only eight irreducible octahedrites in which all CC's are simple circuits. The vertex counts (and symmetries) are 6 (O_h), 12 (O_h, D_{3h}), 14 (D_{4h}), 20 (D_{2d}), 22 (C_{2v}), 30 (O) and 32 (D_{4h}) (Deza and Shtogrin, 2003). The three of octahedral symmetry correspond to cube wrappings with parameters $\{1, 0\}$, $\{1, 1\}$, $\{2, 1\}$, i.e., one example from each of classes I, II and III.

Deza and co-workers also make an intriguing connection between octahedrites and knot theory. Every 4-valent plane graph can be seen as a regular alternating projection of an alternating knot or link (Kawauchi, 1996) and so a weaving is a physical manifestation of an alternating link. Since a wrapping is equivalent to a 4-valent graph $T^\star$, every wrapping corresponds to a weaving. Deza and Shtogrin (2003) catalogue the associations between some small octahedrites and well known objects of knot theory. Clearly, this could give an interesting direction for exploration in practical basketry.

Perhaps the most significant implication of the association between wrappings and octahedrites is that it soon becomes clear that there are many other wrappable polyhedra beyond the simple cube. Any octahedrite $T^\star$ defines a tiling T. The Alexandrov existence and uniqueness theorems (Pak, 2008) guarantee that T can be realised with all faces square as the wrapping of a unique underlying 8-vertex object P, where P is either a polyhedron or a 'doubly covered polygon', i.e., a degenerate polyhedron with just two faces. In general, many non-convex realisations are also possible, corresponding to different distributions of folds in the square faces. In fact, the volume of P can always be increased by taking a non-convex realisation (see Pak, 2008, Theorem 39.4). For practical construction of closed baskets, some of these non-convex realisations may, in fact, be preferable.

Figure 5 gives a complete catalogue of the octahedrites with $n \leq 16$, and Figure 6 gives an example weaving derived from each of the six smallest octahedrites. In each case we have chosen to show the weaving on the Alexandrov polyhedron. A closed basket based on the smallest non-octahedrally symmetric octahedrite (the square antiprism, 8-1 in Figure 5), and woven by Felicity Wood, is shown in Figure 7.

It may be useful to recapitulate the relation between octahedrites of octahedral symmetry and wrappings of the cube. As Figure 2 row (iv) shows, each wrapping on the cube defines an octahedrite graph via the set of mid-lines of all strands. That octahedrite graph will have either full octahedral (O_h) or octahedral rotational (O) point-group symmetry. Conversely, any octahedrite of O_h or O symmetry corresponds to a wrapping of the cube. The convex realisation of the dual of each such octahedrite is

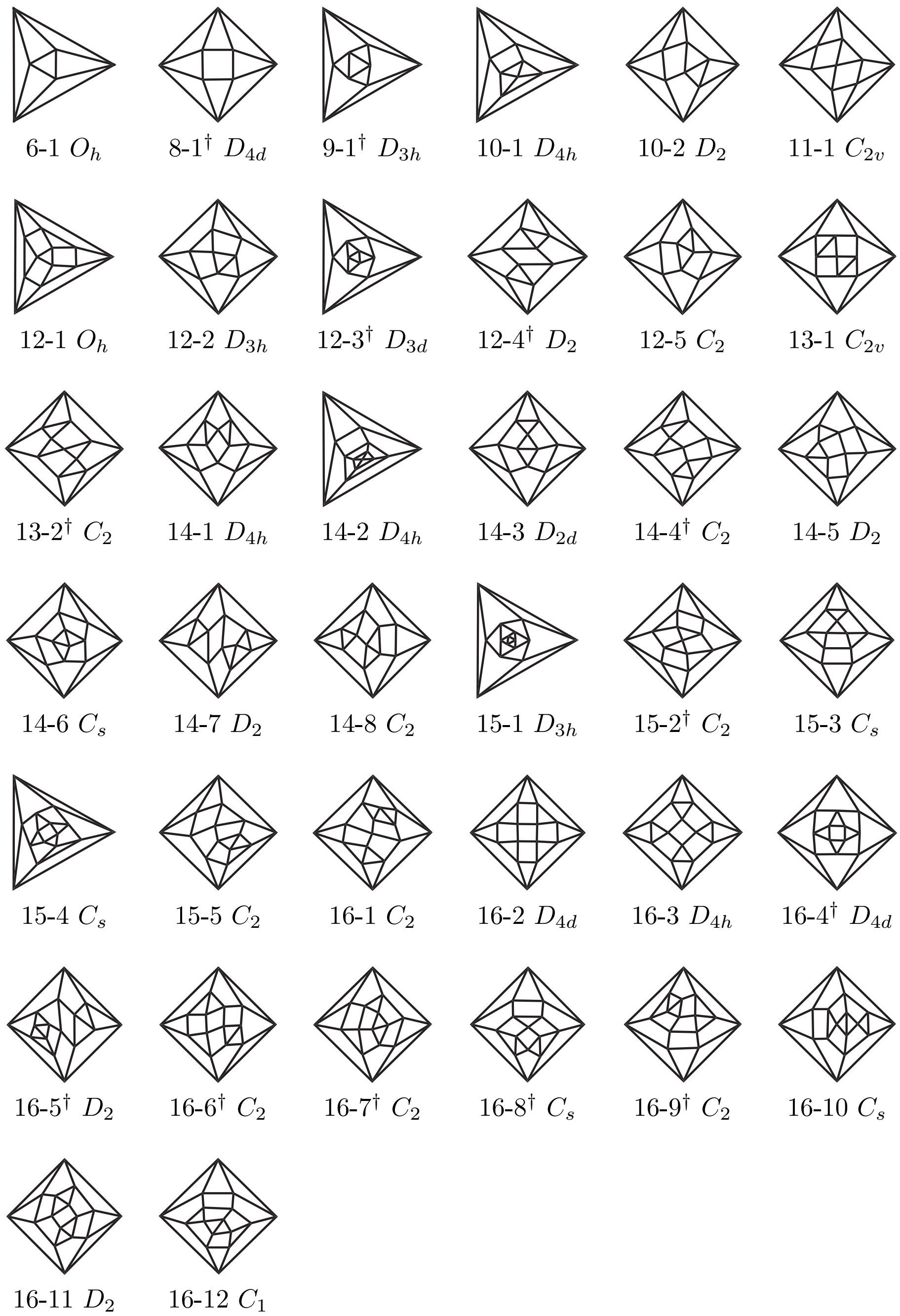
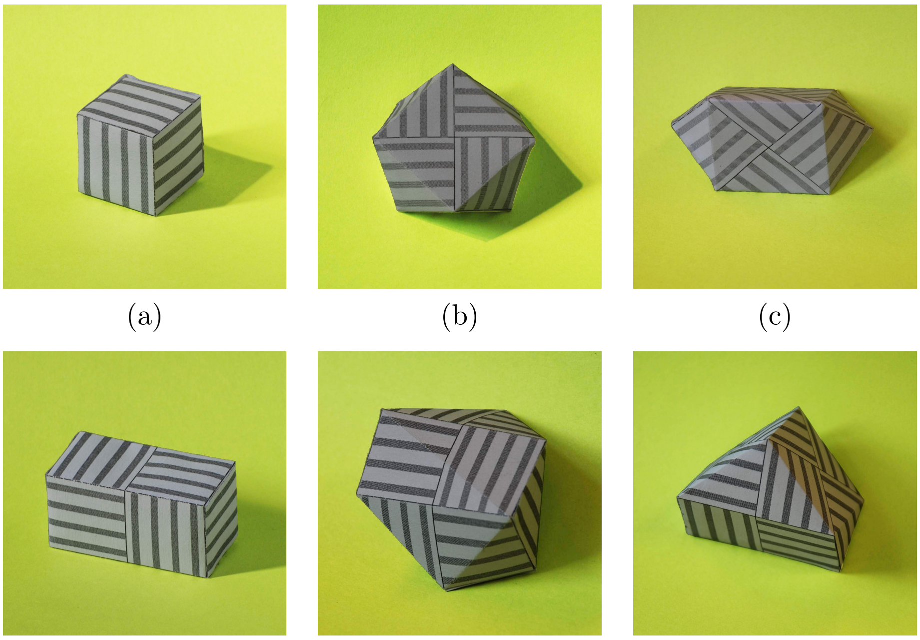


Figure 5: A complete catalogue of octahedrites with $N = 16$ or fewer vertices. The labelling follows Deza and Shtogrin (2003), and includes the vertex number, isomer count and point group. The dagger added to the label indicates an octahedrite with only one central circuit, and hence only one strand in the weaving. The Schlegel diagrams are collated and redrawn from Deza and Shtogrin (2003).



(a) (b) (c)

(d) (e) (f)

Figure 6: Examples of wrappings based on octahedrites. Each strand in a wrapping is shown striped, with four grey stripes running along its length. The wrapped solids are the Alexandrov polyhedra derived from the six smallest octahedrites: (a) $6-1$ O_h; (b) $8-1$ D_{4d}; (c) $9-1$ D_{3h}; (d) $10-1$ D_{4h}; (e) $10-2$ D_2; (f) $11-1$ C_{2v}. The Alexandrov polyhedra are combinatorially equivalent to: (a) the cube; (b) the square antiprism; (c) the bicapped trigonal prism, which is also J14, the 14th Johnson solid (Johnson, 1966); (d) the cube; (e) the snub disphenoid or triangular dodecahedron, J84, (f) the gyrobifastigium, J26. In the numbering used in Read and Wilson (1998) these graphs are: (a) Tc46; (b) Tc249; (c) Tc168; (d) Tc46; (e) Tc301; (f) Tc94. In the numbering of 8-vertex coordination polyhedra given in Britton and Dunitz (1973) they are labelled: (a) 257; (b) 128; (c) 198; (d) 257; (e) 14; (f) 244.

Figure 7: Weaving of the square antiprism. The ribbon forms a single closed strand of 16 unit squares. This closed basket was constructed by Felicity Wood, who also provided the photograph.

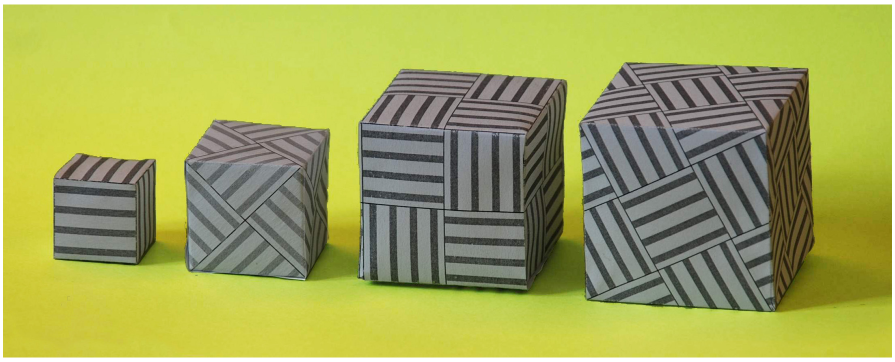


Figure 8: Wrappings of the cube. The tiling T in each case is the dual of an octahedrite with octahedral rotational symmetry: $6-1$ O_h; $12-1$ O_h; 24 O_h; 30 O; .... All four wrappings can be seen as inflations of the first (the unit cube) with $(b,c) = (1,0), (1,1), (2,0), (2,1)$.

a decorated cube, with corners corresponding to triangular faces of the octahedrite graph, and all 4-coordinate vertices of the octahedrite appearing, either on a cube edge, or in a flat region on a face. Figure 8 shows cube wrappings with $(b,c) = (1,0), (1,1), (2,0), (2,1)$, which are derived from 6-, 12-, 24-, and 30-vertex octahedrites, respectively.

Similar reasoning applies to the general, non-octahedrally symmetric octahedrites. By the Goldberg-Coxeter construction (Dutour and Deza, 2004; Deza and Dutour Sikirić, 2007), any octahedrite on $n = n_0$ vertices can be expanded to give an octahedrite on $n = (b^2+c^2)n_0$ vertices. This involves stretching and rotating the net so that each edge-length is multiplied by the inflation factor $\sqrt{b^2+c^2}$. Each octahedrite is therefore the parent of an infinite sequence of inflated versions, and hence it is natural to define *prime* octahedrites as those that are not produced by the inflation of any smaller

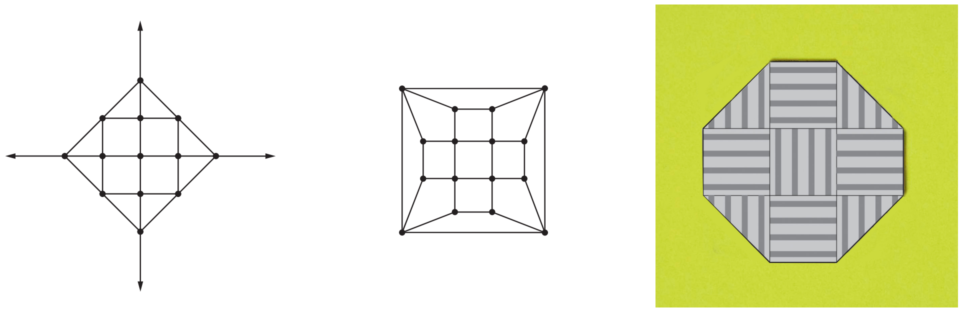
Figure 9: Wrappings of the square antiprism where the triangular faces are right isosceles triangles. The octahedrites that generate these wrappings are the first four Goldberg-Coxeter expansions of the octahedrite 8-1, the first octahedrite with non-octahedral symmetry, The four examples have $(b, c) = (1, 0), (1, 1), (2, 0), (2, 1)$, where the pair of values (b, c) relates to the shorter edge of the triangles.

octahedrite. If the vertex count of a prime octahedrite is inflated by a factor $(b^2 + c^2)$, where $\{b, c\} = \{b, 0\}$ or $\{b, c\} = \{b, b\}$, the enlarged octahedrite has the same symmetries as the prime parent; otherwise it has at least the (proper) rotational symmetries. Each inflation of a prime octahedrite will have a dual with a convex realisation, which will be identical to that of the dual of the prime parent, apart from a geometric scaling by $\sqrt{b^2 + c^2}$. Thus each family could be considered as a sequence of increasingly complex wrappings of the same underlying polyhedron P. Figure 8 shows the first four members of the family where the parent is the smallest octahedrite 6-1, and where P is the cube. Likewise, Figure 9 shows the first four members of the family where the parent is the next smallest octahedrite 8-1, and where P is the square antiprism.

A natural question is: what convex polyhedra can be wrapped? For those polyhedra P that ultimately derive from octahedrites, we have a partial answer. In this case, P must have 8 vertices, and the implication is that there are at most 258 combinatorially distinct P; these comprise the 257 8-vertex polyhedra (Read and Wilson, 1998, chapter 5) and the doubly covered octagon. The examples in Figure 6 show that at least five of the set of 257 polyhedra can be wrapped, some in multiple geometric realisations; the other polyhedral cases have not yet been explored, but we note that the octagon occurs as P for the case 14-1, as shown in Figure 10.

4 Extensions

The family of 'i-hedrites' is a generalisation of the octahedrites: an i-hedrite has $f_2 = 8-i$ digonal faces and $f_3 = 8-2f_2$ triangular faces, with $i = 4, \ldots, 8$



(a) (b) (c)

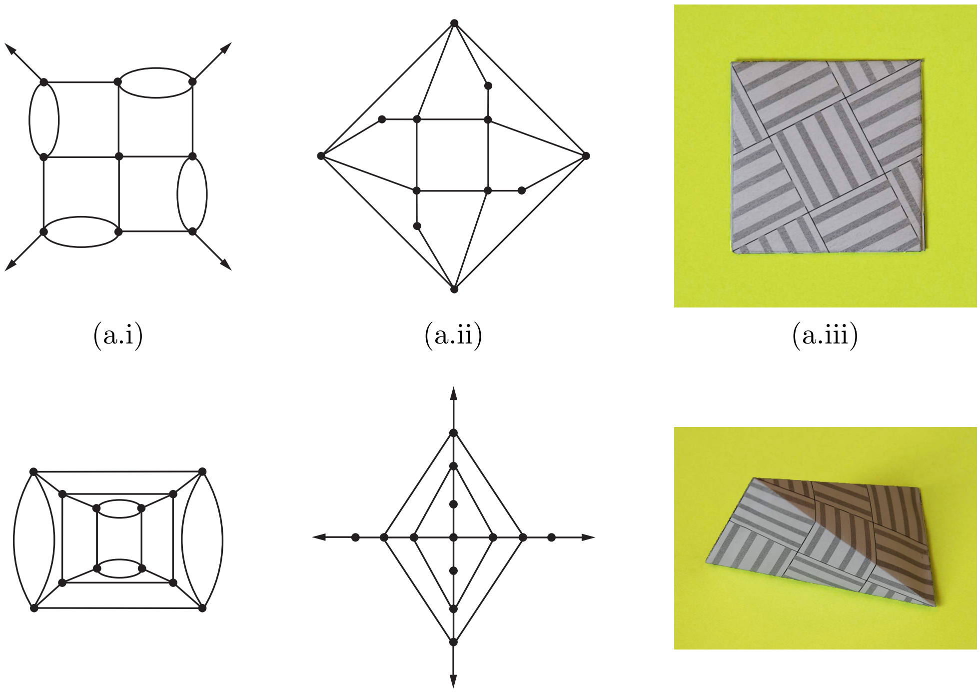
Figure 10: Wrapping an octagon: (a) the octahedrite 14-1, redrawn from Figure 5 to emphasise the full D_{4h} symmetry (with arrows directed to a vertex at infinity); (b) the tiling formed as the dual of the octahedrite; (c) one face of the wrapped polygon.

(Deza et al., 2003). Although not polyhedral, the duals of these graphs generate a square tiling that is an intrinsically convex polyhedral surface, i.e., it has everywhere non-negative curvature, and hence Alexandrov's theorem still applies. Duals of i-hedrites therefore generate wrappings of convex polyhedra (or doubly-covered polygons) in much the same way as octahedrites. Figure 11 gives two examples of i-hedrites, their duals, and the wrapped objects. It is even possible to go a step further in the generalisation of the octahedrites, and allow faces of size 1 (loops) which give univalent vertices in the tiling.

Beyond the octahedrite and i-hedrite classes, those members of the wider class of 4-regular polyhedra that have negative curvature also generate wrappings. The 4-regular polyhedra obey (2), i.e.,

$$1f_3 - 0f_4 - 1f_5 - 2f_6 - \ldots = 8. \tag{3}$$

Those with $f_r > 0$ for some $r > 4$ have a region or regions of negative curvature. The numbers of general 4-regular polyhedra are given by sequence A007022 in Sloane's encyclopedia (Sloane, 2008; Brinkmann et al., 2003)). Figure 12 shows small examples based on dualising polyhedra with pentagonal and hexagonal faces; in this case, non-convex realisations are inevitable because of the negative curvature, and so we have chosen symmetrically crinkled structures to preserve the maximum D_{nd} symmetry of the underlying polyhedra. Examples based on 4-regular polyhedra with even higher symmetries can also be constructed. An icosahedrally symmetric crinkled structure is shown in Figure 13, similar in appearance to the 3-fold woven object presented in Plate D of Pedersen (1981).



(a.i) (a.ii) (a.iii)

(b.i) (b.ii) (b.iii)

Figure 11: Examples of wrapping based on i-hedrites, showing (i) the i-hedrite graph, (ii) the dual, and (iii) the wrapped object, for (a) 10-1 D_4 (with one vertex of the i-hedrite at infinity), and (b) 12-3 D_{2d} (with one vertex of the dual at infinity) The numbering of the i-hedrites follows Deza et al. (2003)).

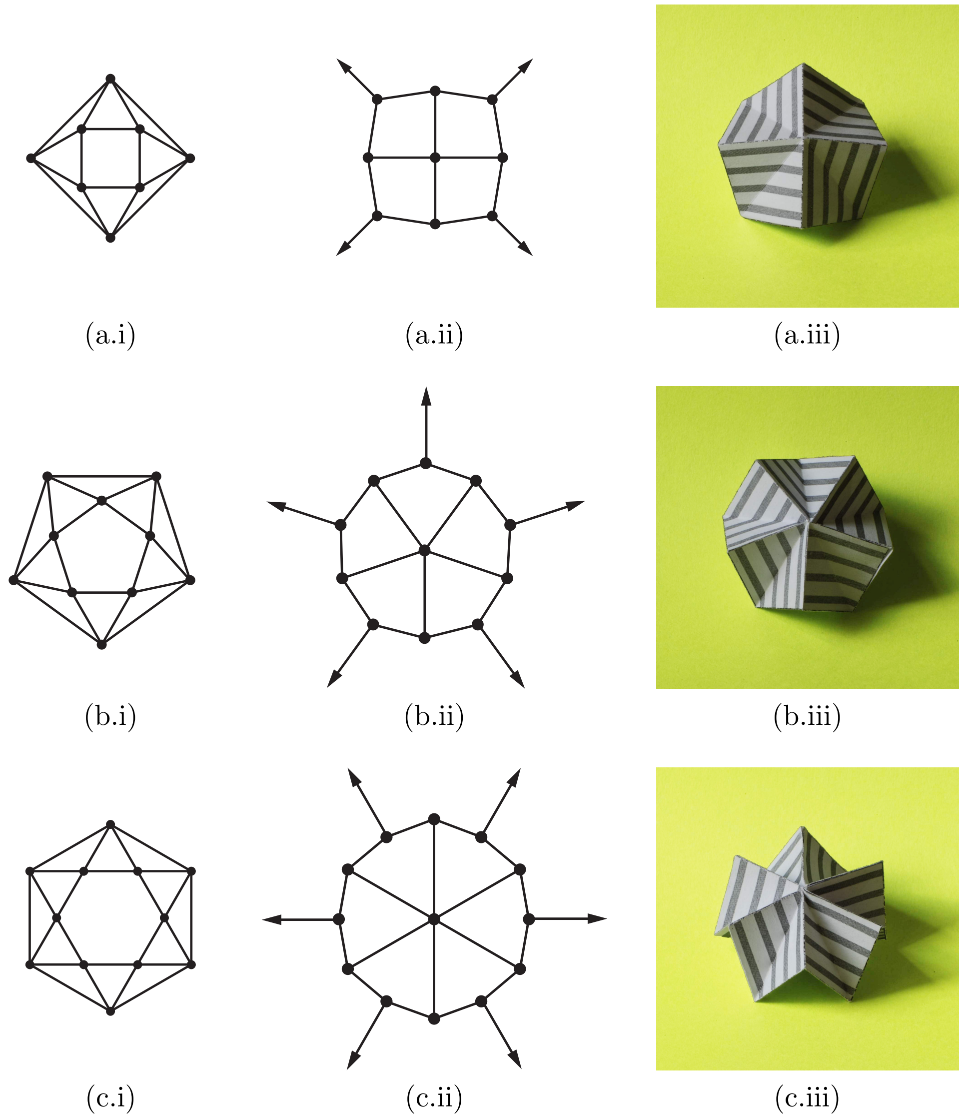


Figure 12: A family of wrapped non-convex polyhedra. The panels show, in column (i), the primal polyhedra, the $[n]$-antiprisms with $n = 4, 5, 6$, i.e., the square, pentagonal and hexagonal antiprisms with $f_3 = 2r, f_r = 2$ for $r = 4, 5, 6$, respectively; in column (ii), their duals, the $[n]$-trapezohedra; in column (iii), the wrapped objects. In the case of the [4]-antiprism (octahedrite 8-1), a convex realisation exists and is shown in Figure 9. For $n \geq 5$, non-convexity makes crinkling unavoidable.

Figure 13: Wrapping of an icosahedrally symmetric non-convex polyhedron. The 4-regular polyhedral graph whose dual defines the tiling is the graph of the icosidodecahedron, with $f_5 = 12$ and $f_3 = 20$.

5 Conclusion

Starting from an analysis of wrappings of one simple highly-symmetric polyhedron, the cube, it has been possible to identify infinite classes of potential wrappings and closed baskets based on convex and non-convex polyhedra. Some open questions remain, such as the characterisation of the 8-vertex polyhedra that may appear in wrappings derived from octahedrites: do all appear as Alexandrov polyhedra of wrappings, and if so, with what symmetry and geometrical realisation?

Weaving on intrinsically curved surfaces presents technical difficulties, e.g., the 'draping' problem in the manufacture of advanced composite components of complex geometry (Hancock and Potter, 2006). Extension of the present considerations, together with modern tow placement machines (Rudd et al., 1999) could help in the development of improved manufacturing techniques. The present findings already provide a pattern-book for future artistic and practical endeavours.

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