Example Questions for Queuing Theory and Markov Chains

Read: Chapter 14 (with the exception of chapter 14.8, unless you are interested) and Chapter 15 of Hillier/Lieberman, *Introduction to Operations Research*

Problem 1: Deduce the formula $L_q = \lambda W_q$ intuitively.

Problem 2: A two-server queueing system is in a steady-state condition and the steady state probabilities are $p_0 = \frac{1}{16}, p_1 = \frac{4}{16}, p_2 = \frac{6}{16}, p_3 = \frac{4}{16}, p_4 = \frac{1}{16}$, $p_n = 0$ if $n > 4$. Calculate

1. L (the expected number of customers in the system) and L_q (the expected number of customers in the queue) (Answer: $L = 2, L_q = 3/8$),

2. the expected number of customers being served (Answer: $13/8$).

Suppose the arrival rate is 2 customers per hour. Calculate

1. W (the mean waiting time in the system) and W_q (the mean waiting time in the queue) (Answer: $W = 1$ hr, $W_q = 3/16$ hrs),

2. $\frac{1}{\mu}$ (the mean service time per customer) (Answer $13/16$ hrs/customer),

3. ρ (the traffic intensity) (Answer: $13/16$).

Problem 3: Verify the following relations for a single server queue:

1. $L = L_q + 1 - p_0$,

2. $L = L_q + \rho$,

3. $p_0 = 1 - \rho$.

Problem 4: Show that the chance that an exponentially distributed variable takes on a value below its mean is more than 60% and that the chance that the value will be below half of the mean is almost 40%.

Problem 5: When are exponentially distributed interarrival (service) times reasonable/unreasonable? Discuss this questions and provide examples.

Problem 6: Suppose a queueing system has two servers, exponential inter-arrival times with mean of 1 hour, and exponential service times with mean of 1 hour per customer. Suppose a customer has just arrived at 12.00 noon.

1. What is the probability that the next arrival will come before 1.00 pm (between 1.00 pm and 2.00 pm, after 2.00 pm)? (Answer: 0.6321, 0.2325, 0.1353, resp.)

2. Suppose no customer arrives before 1.00 pm. What is the probability that the next arrival will come between 1.00 pm and 2.00 pm? (Answer: 0.6321)

3. What is the probability that the number of arrivals between 1.00 pm and 2.00 pm will be zero (one, more than one)? (Answer: 0.3679, 0.3679, 0.2642, resp.)

4. Suppose that both servers are serving customers at 1.00 pm. What is the probability that neither customer will have service completed before 1.01 pm (before 1.10 pm, before 2 pm)? (Answer: 0.9672, 0.7165, 0.1353, resp.)

Problem 7: A dental surgery has two operation rooms. The service times are assumed to be independent, exponentially distributed with mean 15 minutes. Andrew arrives when both operation rooms are empty. Bob arrives 10 minutes later while Andrew is still under medical treatment. Another 20 minutes later Caroline arrives and both Andrew and Bob are still under treatment. No other patient arrives during this 30-minute interval.

1. What is the probability that Andrew will be ready before Bob?(Answer: 1/2)

2. What is the probability that Caroline will be ready before Andrew?(Answer: 1/4)

3. What is the probability that Caroline will be ready before Bob? (Answer: 1/4)

4. Determine the distribution function of the waiting time in the system for Caroline. Find the mean and variance. (Answer: $F(t) = 1 - 2e^{-\lambda t} + e^{-2\lambda t}$, Exp.=22.5 minutes, Var.=$(16.8)^2$ minutes^2)

Problem 8:

a) Prove that the exponential distribution has both the lack of memory and the minimum property.

b) What are the characteristic properties of the Poisson input process? Discuss their limitations by giving examples from practical queueing systems.

c) Suppose there are 2 types of customers arriving at the same server according to independent Poisson input processes. Show that the aggregated arrival process (of customers without regard of type) is a Poisson process.

Problem 9:

Complicated arguments often involve series of implications of the form "A_0 therefore A_1 therefore A_2 therefore...", where the A_i's are propositions. The truth of any proposition in such an argumentation chain depends on the validity of the foregoing implications. Often such implications are not watertight but only true with a high probability. For the purpose of this question suppose if A_n is true then A_{n+1} is true with a 90% chance while if A_n is false then A_{n+1} is true or false with equal probability ("ex falso quodlibet").

1. Formulate the change of the truth values of the propositions as a Markov chain and give the transition matrix

2. What does the Markov property mean in this setting? Discuss this assumption. When does it apply?

3. What is the probability that A_4 is true, provided A_0 was true with 50% probability (0.8248).

4. Calculate the stationary probabilities and interpret them $((1/6, 5/6))$.

5. If an adversary checks the argumentation chain and finds one false statement, then she would argue that the argumentation chain is invalid and would reject any further conclusions. We may therefore say that an argumentation chain breaks down if

the first false statement is made. Change the Markov chain to model this situation.

6. Calculate the probability that the argumentation chain is still valid after the fourth statement, provided the initial statement was true (0.6561).

7. Calculate the stationary probabilities for the new Markov chain and interpret them ($(1,0)$).

8. Calculate the probability that, starting from a true proposition, the argumentation chain breaks down after precisely 3 iterations (8.1%).

9. The adversary of part 5 will only examine the argumentation chain if she can be 50 % sure that she will find a false proposition. How long does the argumentation chain have to be in order for this to be the case, provided A_0 is true. Use a spreadsheet to make your calculations. (7 implications)

10. Calculate the expected number of statements until the argumentation chain breaks down if the initial statement is true. (10 implications)

11. Give a formula for the expected number of statements until the argumentation chain breaks down in terms of the probability p that the next statement holds, provided the current statement is true.

Additional relevant solved problems can be found in Chapters 19-24 of Richard Bronson, *Schaum's outline of theory and problems of operations research* (to be available on the G12 shelf soon).