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THE RICOCHET OF SPHERES AND CYLINDERS FROM THE SURFACE OF WATER

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Summary—A theory of ricochet is proposed which permits the effect of projectile spin to be accounted for. This effect is not explained by previous theories. The critical angles for ricochet for a sphere and for a spinning cylinder are calculated, and the theory of the spinning cylinder is applied to the Barnes Wallis "bouncing bomb" developed during World War II.

"Experience, therefore, is the best guide in the doctrine of either throwing bombs or shooting balls; for the theory, of which we have given here but a small part, can carry the young engineer but a short way, and indeed it is rather an object of amusement than utility."

OLIVER GOLDSMITH
"*A Survey of Experimental Philosophy*", 1776.

NOTATION

- a radius of sphere or cylinder
- D hydrodynamic drag force on cylinder
- dA element of area on surface of sphere or cylinder
- g acceleration due to gravity
- l length of cylinder
- L hydrodynamic lift force on sphere or cylinder
- m mass of projectile
- p hydrodynamic pressure
- R $= L/D$
- t time
- v velocity of projectile's centre of mass
- v' velocity of element of surface of cylinder relative to the water
- x, y cartesian co-ordinates of centre of sphere or axis of cylinder
- (a, ϕ, ψ) spherical polar co-ordinates of point on surface of sphere
- (a, ϕ) cylindrical co-ordinates of point on surface of cylinder
- α angle between v' and the surface of the cylinder
- β angle between v and the outward normal from dA
- γ angle between v and v'
- θ_c critical angle of incidence for ricochet
- ρ density of water
- ρ' density of projectile
- σ specific gravity of projectile $= \rho'/\rho$
- ϕ_0 co-ordinate of point on sphere or cylinder corresponding to level of undisturbed water surface
- ϕ_w co-ordinate of point on sphere or cylinder corresponding to extent of wetted area
- ω axial angular velocity of spinning cylinder

1. INTRODUCTION

THE REBOUND or ricochet of solid projectiles striking a water surface at small angles of incidence has been exploited in military engineering for over 200 yr. Early investigations of the phenomenon, reviewed by Johnson and Reid,[1] showed empirically that the critical angle of incidence, θ_c, above which ricochet of spherical shot from water will not occur, is given by;

$$\theta_c = 18/\sqrt{(\sigma)} \quad \text{(degrees)} \qquad (1)$$

where σ is the specific gravity of the projectile. In World War II, ricochet was used by Sir Barnes Wallis in his design of a weapon, the "bouncing bomb" for attacking dams. Wallis' design was novel in that back-spin was applied to the bomb before release from the aircraft. The spin increased the critical angle for ricochet and "the velocity and range of a bomb dropped from a low height... "*it would either bounce over the defensive netting or else have sufficient momentum to break through*".[2]

Johnson and Reid[1] have recently applied the theory of Birkhoff *et al.*,[3] to the ricochet of a spherical projectile from water and showed that the theory is in good agreement with equation (1), and with the experimental results of Richardson.[4] However, Birkhoff's theory does not predict any effect of the spin of the sphere on its ricochet; it is therefore incapable of explaining the benefit which Wallis appears to have gained by imparting spin.

The Birkhoff theory assumes that the pressure acting on a surface element whose outward normal makes an angle $\beta \leq \pi/2$ with the line of forward motion is $\frac{1}{2}\rho v^2 \cos^2 \beta$, where v is the current speed of the projectile, and ρ the density of the water, and that this pressure applies only to parts of the sphere below the undisturbed surface of the water. Both these assumptions are physically unrealistic; a more satisfactory theory is proposed in the next section. This theory enables the effect of projectile spin to be predicted, and will be applied to the case of a sphere without spin for comparison with equation (1). Although experimental bombs studied by Wallis were spherical, the projectiles used in the raid against the dams were cylindrical, and data for them are readily available.* The analysis will therefore also be applied to a spinning cylindrical projectile.

2. RAYLEIGH'S PRESSURE FORMULA

Birkhoff's model assumes that the pressure acting on an element of surface at an angle β to a stream of non-viscous fluid, of density ρ and velocity v, is $\frac{1}{2}\rho v^2 \cos^2 \beta$. In fact, the pressure on a lamina under these conditions varies more nearly as $\cos \beta$; the exact relationship for the mean pressure, p, was shown by Rayleigh[8] to be

$$p = \frac{\pi \cos \beta}{4 + \pi \cos \beta} \rho v^2. \tag{2}$$

"*The fact that the resistance to the broadways motion of a lamina through still fluid can be increased enormously by the superposition of an edgeways motion is of great interest*" (Rayleigh, 1876).[8]

Although Rayleigh's result was derived for a plane lamina, and is only strictly true for steady-state laminar flow, it will be assumed in the following sections that the pressure on any element of a curved surface in contact with the water is given by this equation. This is clearly only an approximation to the complex pressure distribution acting on a partially submerged projectile, but it will be argued below that equation (2) gives a better representation of the true forces than Birkhoff's equation.

Figure 1 shows the wetted area of a sphere at various

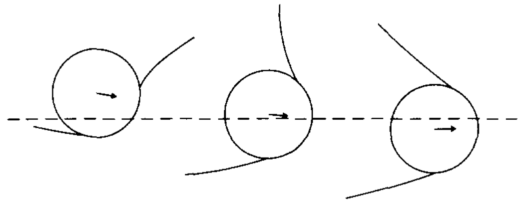

FIG. 1. The shape of the water surface around a sphere at various stages of oblique impact, after Richardson.[4]

depths of immersion during oblique impact on water; these diagrams were taken from those of Richardson.[4] From Fig. 1 it appears that the water pressure must act over a considerably greater area than that assumed by Birkhoff *et al.*;[3] in this work it is proposed that ϕ_w, the angular extent of the wetted area, over which the pressure acts, is equal to $2\phi_0$ for $0 \leq \phi_0 \leq \pi/2$ and $\phi_w = \pi$ for $\phi_0 > \pi/2$, as shown in Fig. 2.

3. SPHERICAL PROJECTILE WITH NO SPIN

The ricochet of a projectile is caused by the vertical component of the pressure distribution resulting from its

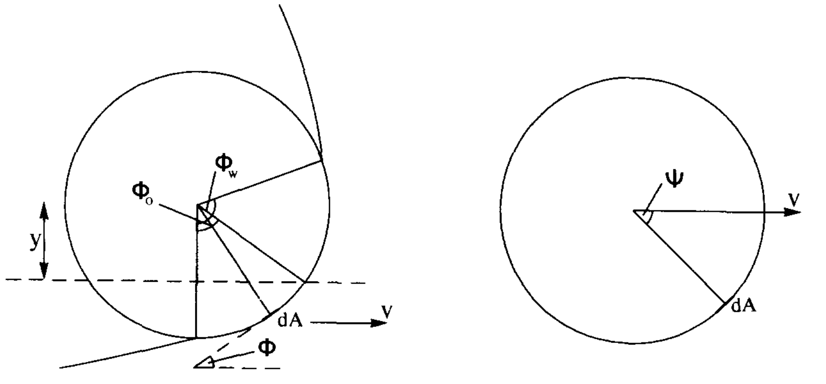

FIG. 2. Elevation and plan views of a partially submerged sphere, defining the angles ϕ, ϕ_0, ϕ_w and ψ.

motion, partially submerged, through the water. We shall here calculate the lift force on a sphere of radius a whose centre is at a height y above the undisturbed water surface (Fig. 2). From the assumptions of the previous section, the pressure will act over the whole wetted area shown in Fig. 2. The water velocity normal to the element of the sphere's surface dA will be $v \sin \phi \cos \psi$. The pressure p acting on dA is therefore

$$p = \frac{\pi \sin \phi \cos \psi}{4 + \pi \sin \phi \cos \psi} \rho v^2. \tag{3}$$

*Morpurgo[2] shows a photograph of an oblate spherical bomb. However, this was clearly an experimental version, since his text and other references[5,6] indicate that the bombs dropped in the raid on the dams (project Downwood) were in fact cylindrical. A photograph showing a definitely cylindrical bomb in its mounting beneath a Lancaster bomber has been published.[6] Another smaller bomb (codenamed Highball) developed for naval use in the Mosquito aircraft and employing exactly the same principles of ricochet assisted by backspin was an oblate sphere.[7]

This expression may be rendered more tractable by making an approximation. The pressure will be integrated over the wetted surface; ϕ will vary between 0 and a maximum of π, ψ between $-\pi/2$ and $\pi/2$. The denominator in equation (3) will therefore vary between 4.0 and 7.1; no great inaccuracy will be introduced by replacing it by the intermediate value of 5.0. The lift force on the area dA is therefore

$$dL = \frac{\pi}{5}\rho v^2 \sin\phi \cos\psi \cos\phi \, dA. \tag{4}$$

Since $dA = a^2 \sin\phi \, d\phi \, d\psi$, the total lift on the sphere shown in Fig. 2 is

$$L = \frac{\pi}{5}\rho v^2 a^2 \int_0^{\phi_w}\int_{-\pi/2}^{\pi/2} \sin^2\phi \cos\phi \cos\psi \, d\psi \, d\phi$$

$$= \frac{2\pi}{15}\rho v^2 a^2 \sin^3 \phi_w. \tag{5}$$

Ignoring gravitational forces on the sphere, its equation of motion vertically is

$$m\frac{d^2y}{dt^2} = L \tag{6}$$

where m is the mass of the sphere. Assuming that the impact and rebound angles are small the velocity, $v = (dx/dt)$, of the sphere may be assumed constant throughout the impact. So

$$\frac{d^2y}{dt^2} = \frac{dx}{dt}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right) = v^2\frac{d^2y}{dx^2}. \tag{7}$$

Substituting in equation (6) for L and the mass of the sphere (density ρ') and using equation (7),

$$\frac{d^2y}{dx^2} = \frac{1}{10a}\frac{\rho}{\rho'}\sin^3\phi_w. \tag{8}$$

From our initial assumptions, $\phi_w = 2\phi_0$, so $\sin\phi_w = 2\sin\phi_0\cos\phi_0$ and

$$\frac{d^2y}{dx^2} = \frac{8}{10a}\frac{\rho}{\rho'}\sin^3\phi_0\cos^3\phi_0 = \frac{8}{10a^7}\frac{\rho}{\rho'}y^3(a^2-y^2)^{3/2}. \tag{9}$$

Integrating equation (9) and multiplying through by $2dy/dx$,

$$d\left(\left(\frac{dy}{dx}\right)^2\right) = \frac{8}{5a^7}\frac{\rho}{\rho'}y^3(a^2-y^2)^{3/2}\,dy. \tag{10}$$

When impact occurs at the critical angle θ_c, dy/dx at the beginning of the impact is equal to θ_c (for small θ_c). The lift on the sphere becomes zero when $\phi_w = \pi$, i.e., when $y = 0$ and when $dy/dx = 0$. Therefore,

$$\int_{\theta_c^2}^{0} d\left(\left(\frac{dy}{dx}\right)^2\right) = \frac{8}{5a^7}\frac{\rho}{\rho'}\int_a^0 y^3(a^2-y^2)^{3/2}\,dy \tag{11}$$

i.e.,

$$\theta_c^2 = \frac{8}{5a^7}\frac{\rho}{\rho'}\left(\frac{a^7}{5}-\frac{a^7}{7}\right) = \frac{16}{175}\frac{\rho}{\rho'}. \tag{12}$$

The critical angle is therefore found to be

$$\theta_c = 0{\cdot}302/\sqrt{(\sigma)} \quad \text{(rad.)}. \tag{13}$$

This value of θ_c, corresponding to $17{\cdot}3/\sqrt{(\sigma)}$ deg., agrees well with the empirical relationship (equation (1)) and with the relationship deduced by Johnson and Reid[1] from Birkhoff's pressure law ($\theta_c = 17{\cdot}5/\sqrt{(\sigma)}$ deg.). A small error in the numerical value of the constant in equation (13) would undoubtedly be revealed by an accurate numerical treatment, in which the denominator in equation (3) is not assumed constant.

4. CYLINDRICAL PROJECTILE WITH SPIN

The forces acting on a cylindrical projectile, possessing back-spin about its axis with angular velocity ω and moving in a plane perpendicular to its axis, will now be considered. The velocity of the water relative to an element of surface dA, v', may be deduced from Fig. 3

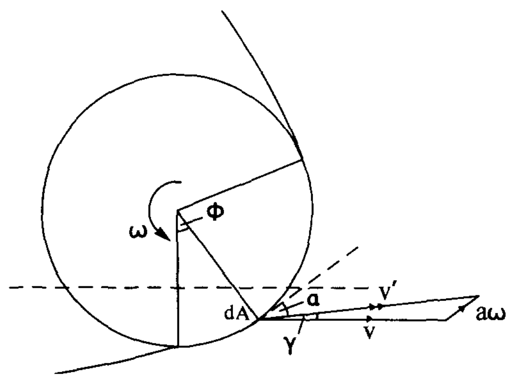

FIG. 3. Cylindrical projectile with back-spin, partially submerged. The velocity of the surface element dA relative to the water is v'. Angles ϕ_0 and ϕ_w are as defined in Fig. 2.

which shows the direction of spin. The angles ϕ, ϕ_0 and ϕ_w are as shown in Fig. 2. The lift force acting on the element is

$$dL = \frac{\pi\sin\alpha}{4+\pi\sin\alpha}\rho v'^2\cos\phi\,dA \tag{14}$$

where

$$v'^2 = (v + a\omega\cos\phi)^2 + (a\omega\sin\phi)^2 \tag{15}$$

and

$$\sin\alpha = \sin\phi\cos\gamma - \cos\phi\sin\gamma. \tag{16}$$

We assume that $a\omega \ll v$ and neglect terms in $a^2\omega^2/v^2$. The angle γ is therefore small,

$$v'^2 \simeq v^2 + 2a\omega v\cos\phi \tag{17}$$

$$\sin\gamma \simeq \gamma \simeq \frac{a\omega}{v}\sin\phi \tag{18}$$

and

$$\sin\alpha \simeq \sin\phi - \frac{a\omega}{v}\sin\phi\cos\phi. \tag{19}$$

For a cylinder of length l, $dA = al\,d\phi$, and making the approximation that the denominator in equation (14) is equal to 5·0,

$$dL = \frac{\pi}{5}\rho al(v^2 \sin\phi + a\omega v \sin\phi \cos\phi)\cos\phi \, d\phi. \quad (20)$$

Exactly following the arguments of the previous section, the total lift for partial immersion is equal to $mv^2 \, d^2y/dx^2$:

$$L = \frac{\pi}{5}\rho alv^2\left(\frac{1}{2}\sin^2\phi_w + \frac{1}{3}(1-\cos^3\phi_w)a\omega/v\right) \quad (21)$$

$$= \pi\rho' a^2 lv^2 \, d^2y/dx^2. \quad (22)$$

Hence, after putting $\phi_w = 2\phi_0$ and substituting for ϕ_0 in terms of y, as before, we have

$$\theta_c^2 = \frac{4}{5}\frac{\rho}{\rho'}\frac{1}{a}\int_0^a \left(y^2/a^2 - y^4/a^4 + \frac{1}{3}(1-4y^6/a^6 + 6y^4/a^4 - 3y^2/a^2)\frac{a\omega}{v}\right) dy$$

$$= \frac{8}{75}\frac{\rho}{\rho'}\left(1 + \frac{11}{7}\frac{a\omega}{v}\right). \quad (23)$$

The critical angle is therefore given by

$$\theta_c = 0{\cdot}327(1 + 1{\cdot}6a\omega/v)^{1/2}/\surd(\sigma) \quad \text{(rad.)} \quad (24)$$

or

$$\theta_c = 18{\cdot}7(1 + 1{\cdot}6a\omega/v)^{1/2}/\surd(\sigma) \quad \text{(deg.)}. \quad (25)$$

For a cylinder without spin, the critical angle is thus very close to that for a sphere of the same density. A moderate amount of spin will cause a significant change in θ_c; for example, a value of $a\omega/v = +(1/3)$ will increase θ_c from $18{\cdot}7/\surd(\sigma)$ deg. (no spin) to $23{\cdot}1/\surd(\sigma)$ deg. A similar amount of top-spin ($a\omega/v = -(1/3)$) would decrease the critical angle to $12{\cdot}9/\surd(\sigma)$ degrees. While, according to this theory, there would be no lift force acting on a totally submerged cylinder without spin, it is instructive to note that when back-spin is applied a submerged cylinder experiences a lift (from equation (21)) of

$$L = \frac{2\pi}{15}\rho a^2 lv\omega \quad (26)$$

where the positive sense of ω is shown in Fig. 3. The criteria for critical incidence applied in deducing equations (11) and (23) are therefore not strictly true, since the lift force does not become zero when $\phi_w = \pi$; the value of θ_c expressed in equation (25) forms a lower bound to the true value (or, in the case of top-spin, an upper bound).

The exact influence of an increase in lift force on the rebound angle and velocity of the projectile can only be deduced by a numerical treatment of the equation of motion of the body. However, an increase in the ratio;

$$R = \text{lift force}/\text{drag force} \quad (27)$$

must lead to a decrease in the energy dissipated by the drag force and hence to an increase in rebound velocity. The drag force, D, on a spinning cylinder is the integral of the horizontal components of the pressure over the wetted surface. This may be shown to be;

$$D = \frac{\pi}{5}\rho alv^2\left(\frac{1}{2}\phi_w - \frac{1}{4}\sin 2\phi_w + \frac{1}{3}\frac{a\omega}{v}\sin^3\phi_w\right). \quad (28)$$

Taking the lift force, L, from equation (21), $R = L/D$. For $\phi_w = \pi/2$, and no spin, $R = 0{\cdot}637$. For $\phi_w = \pi/2$ and $a\omega/v = +(1/3)$, $R = 0{\cdot}682$. The spin therefore results in a 7% increase in the lift: drag force ratio. As the depth of immersion, and hence ϕ_w, increases, the gain in R resulting from back-spin increases further until at total immersion, $\phi_w = \pi$, $R = 0$ for $a\omega/v = 0$ and $R = 0{\cdot}141$ for $a\omega/v = +(1/3)$.

5. DISCUSSION

Application of the Rayleigh pressure formula (equation (2)) to the case of a spherical projectile yields a critical angle in good agreement with the experimentally determined value and with that predicted by Birkhoff's theory.[1] In the absence of published data on the ricochet of cylinders, we can only compare the value of θ_c derived here (equation (25)) with that deduced from Birkhoff's theory. Following the method of Johnson and Reid[1] it may be shown that a critical angle

$$\theta_c = 0{\cdot}354/\surd(\sigma) \quad \text{(rad.)} = 20{\cdot}3/\surd(\sigma) \quad \text{(deg.)} \quad (29)$$

would be predicted for a cylinder, with no dependence of ricochet on spin.

5.1 *The Wallis "bouncing bomb"*

The analysis presented in section 4 indicates that an increased critical angle and rebound velocity should be gained by imparting back-spin to a cylindrical projectile. These gains were clearly known to Wallis as a result of his (unpublished) experiments.[2] The bombs dropped in the air-raids on the Ruhr dams were cylindrical, 1·27 m dia. and 1·52 m long, weighing 4·20 Mg.[2] Their specific gravity was therefore 2·17, and from equation (25) the critical angle for ricochet, with no spin applied, would have been *ca.* 12 deg. The bombs were released from aircraft with a horizontal velocity of 110 ms^{-1} from a height of 18 m, spinning at 52 rad s^{-1}.* Assuming no air resistance, the impact angle was therefore *ca.* 10 deg.; the effects of air resistance, wind and roughness of the water surface might easily have prevented ricochet on the initial impact, had back-spin not been applied. The effect of the applied spin, with $a\omega/v \simeq 1/3$, was to increase the critical angle to *ca.* 16 deg. and thereby greatly increase the chance of successful ricochet. The spin would also have increased the rebound velocity and hence the range of ricochet, as discussed above.

Two further mechanisms exist by which the range of a bouncing projectile may be increased by the application of back-spin: aerodynamic lift (the Magnus effect) and the effect of spin on aerodynamic drag. Lift and drag forces acting on spheres and cylinders spinning in air have been studied[12] at Reynolds numbers up to $\sim 10^5$. Unfortunately no data are available at the Reynolds number of $\sim 9 \times 10^6$ which would apply to Wallis' bomb travelling at 110 ms^{-1} in air. However, for Reynolds numbers of *ca.* 10^5 and $a\omega/v < \sim 0{\cdot}5$, spheres and cylinders experience zero or negative lift.[12] At higher values of $a\omega/v$ the lift coefficient rises rapidly. The drag coefficient is found to fall slightly as $a\omega/v$ increases from 0 to ~ 1, but increases again rapidly for $a\omega/v > 1$. These aerodynamic effects are clearly complex and not easily analysed; any estimate of the aerodynamic influence of spin on Wallis' bomb must

*All sources agree that the bombs were dropped from a height of 60 ft. They differ on the aircraft's velocity, giving figures of 220 mph,[6,9] 232 mph,[10] 240 mph[11] and 250 mph.[2] A groundspeed of 250 mph, which gives the lowest impact angle, has been assumed here.

therefore be qualitative. It would seem that before the initial impact, when $a\omega/v$ was $\sim 1/3$, the back-spin would have caused zero or even negative Magnus lift and the aerodynamic drag would have been only marginally decreased. After the initial bounce, however, $a\omega/v$ may well have increased, causing an appreciable Magnus lift force and possibly a lower air drag. Both would have contributed to a lengthening of the range of the bomb. It is clear therefore that while these aerodynamic forces may have become important after the initial ricochet of the bomb, the major effect of spin on at least the first bounce will have been though the hydrodynamic mechanism discussed above.

6. CONCLUSIONS

The theory due to Birkhoff *et al.*,[3] previously applied to the problem of ricochet from water has been discarded in favour of a pressure law described by Rayleigh.[8] The Rayleigh formula is applied here to a spherical projectile and to a spinning cylindrical projectile. This theory predicts similar critical angles for ricochet for a sphere and for a cylinder without spin.

The Rayleigh pressure distribution permits the influence of projectile spin on ricochet to be deduced, and predicts the increased critical angle, rebound velocity and range of a projectile with back-spin, utilized by Barnes Wallis in the design of his bouncing bomb.[2] This effect cannot be accounted for by the Birkhoff theory.[1]

Note added in proof. Sir Barnes Wallis has recently kindly supplied me with a copy of his note "*Air Attack on Dams*" (1943). In this paper he suggests that the benefits of back-spin result from aerodynamic lift forces, and does not consider the influence of spin on the hydrodynamic forces acting while the bomb is in contact with the water. Unfortunately, no further comparison of the theory presented above with Wallis' experimental data is possible, since his original data were accidentally destroyed in 1968. The contribution made by the hydrodynamic mechanism described above must therefore remain unresolved until experimental tests can be made.

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